AUTOMORPHIC SYMBOLS, p-ADIC L-FUNCTIONS AND ORDINARY COHOMOLOGY OF HILBERT MODULAR VARIETIES

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ABSTRACT. We introduce the notion of automorphic symbol generalizing the classical modular symbol and use it to attach very general p-adic L-functions to nearly ordinary Hilbert automorphic forms. Then we establish an exact control theorem for the p-adically completed cohomology of a Hilbert modular variety localized at a suitable nearly ordinary maximal ideal of the Hecke algebra. We also show its freeness over the corresponding Hecke algebra which turns out to be a universal deformation ring. In the last part of the paper we combine the above results to construct p-adic L-functions for Hida families of Hilbert automorphic forms in universal deformation rings of Galois representations.

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Introduction

Throughout this paper F will denote a totally real number field of degree $d > 1$ and ring of integers $\mathfrak{o}$. Let I be the set of its infinite places. Let p be a prime number and let $\mathcal{O}$ be the ring of integers of a p-adic field E which is sufficiently large (to be specified later). Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers in $\mathbb{C}$ and let us fix an embedding of $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}}_p$.

0.1. **Automorphic symbols and p-adic L-functions.** Modular symbols arise classically as first homology classes of modular curves, represented by geodesics connecting rational points on the boundary of the upper half plane. In general, the closure of the image of a standard Shintani cone in the compactified Hilbert modular variety yields a d-cycle, hence an element in the d-th homology group that we call a Manin-Oda modular symbol (see Oda's book [Od] for the case of Hilbert modular surfaces). The relations between those symbols remain mysterious even in the simplest case of a Hilbert modular surface.

In the spirit of Tate's thesis, we define in §1 an "adelic" counterpart of the Manin-Oda modular symbols, called *automorphic symbols.* Their importance (in particular their non-vanishing) will become visible in §2 where they are related to L-values. When $F \neq \mathbb{Q}$ the presence of units and non-trivial class groups complicates substantially the evaluation of cohomology classes on automorphic symbols. Those difficulties are addressed and overcome in §1.5 which makes it a key part of our construction. Our definition of automorphic symbol is group-theoretic and can be easily transposed to other groups than GL_2.

Analytic p-adic L-functions for classical modular forms have been extensively studied by Mazur, Manin, Vishik, Amice-Velu *et al.* by means of modular symbols (we refer to [MTT] for a complete exposition of their results). For Hilbert modular forms, besides the precursory work of Manin [Ma], the question has been studied by Dabrowski [Da], Panchishkin [Pa] and Mok [M1] using Rankin-Selberg convolution, and more recently by Januszewski [Ja]. Nevertheless, prior to the present article several aspects of the theory remained unavailable in the Hilbert modular setting, namely:

- allowing horizontal twists, that is, twists by characters ramified at some finite set Σ of primes of F not dividing p;

- allowing nearly ordinary forms; note that a nearly ordinary Hilbert automorphic form is not necessarily a twist of an ordinary one by a Hecke character unramified outside $p\infty$;
- allowing forms that are not necessarily new, namely forms whose Euler factors are 1 at primes in Σ;
- in [Da, Pa] the level is prime to p, whereas in [M1] the weight is parallel and in [Ja] the class number is not divisible by p.

We use the flexibility of our construction of automorphic symbols to prove the following theorem (see theorem 2.11 for more precise statement).

Theorem 0.1. *Let π be a cohomological cuspidal automorphic representation of GL_2 /F whose weight (w, w_0) is critical (see definitions 1.16 and 2.1). Assume that π is nearly ordinary at all places dividing p (see definition 2.2). Fix a finite set Σ of primes not dividing p and let $F^{(p\Sigma)}$ be the maximal abelian pro-p extension of F unramified outside $p\Sigma\infty$. Then there exists a p-adic L-function $L_p^\Sigma(\pi) \in \mathcal{O}[[\mathrm{Gal}(F^{(p\Sigma)}/F)]]$, uniquely determined by the following interpolation property: for every finite order character $\phi : \mathrm{Gal}(F^{(p\Sigma)}/F) \to \mathcal{O}^\times$ the image of $L_p^\Sigma(\pi)$ by the resulting homomorphism $\mathcal{O}[[\mathrm{Gal}(F^{(p\Sigma)}/F)]] \to \mathcal{O}$ equals:*

$$\frac{L^{(p\Sigma)}(\pi \otimes \phi, 1)\Gamma(\pi, 1)}{\Omega_\pi^\Sigma} \prod_{v|p\Sigma} Z_v,$$

where $L^{(p\Sigma)}(\pi \otimes \phi, s)$ denotes the L-function whose Euler factors are 1 at primes dividing $p\Sigma$, Ω_π^Σ is an Archimedean period (see definition 2.9) and Z_v are local terms.

Moreover, if $(w, w_0 - 2)$ is critical too, then the automorphism $[a] \mapsto (\chi\omega^{-1})(a)[a]$ of $\mathcal{O}[[\mathrm{Gal}(F^{(p\Sigma)}/F)]]$ sends $L_p^\Sigma(\pi)$ to $L_p^\Sigma(\pi \otimes |\cdot|_{\mathbb{A}}\omega^{-1})$, where χ denotes the p-adic cyclotomic character and ω is the Teichmüller character.

One reason to look for p-adic L-functions of Hilbert automorphic forms in that generality is for the construction of p-adic L-functions of Hida families of Hilbert automorphic forms (see theorems 0.2 and 0.3).

0.2. **Ordinary cohomology.** To any cohomological cuspidal automorphic representation π of GL_2 /F, one can attach a two-dimensional p-adic representation $\rho_{\pi,p}$ defined over a p-adic field. In [H1, H3] Hida developed the theory of nearly ordinary families of Hilbert automorphic forms and proved a control theorem for the nearly ordinary p-adic Hecke algebras, which allowed him to construct two-dimensional Galois representations over these algebras interpolating the $\rho_{\pi,p}$'s when π varies in a Hida family. Hida's proof relies, via the Jacquet-Langlands correspondence, on control theorems for the nearly ordinary cohomology of Shimura varieties of dimension zero or one, coming from quaternion algebras over F which are totally definite, or indefinite but yielding Shimura curves. The introduction of [H1] concludes with the hope that those results could be extended to other quaternionic Shimura varieties. In §3 we extend Hida's results to the case of the indefinite quaternion algebra $M_2(F)$ by proving an exact control theorem for the nearly ordinary cohomology of Hilbert modular varieties (see theorem 3.8). The proof proceeds by specialization to a well chosen finite level and weight where the main results of [D1, D2] can be applied.

The last decade has seen the emergence of p-adic and mod p Langlands programs. Although they are widely conjectural for a general reductive algebraic group over a number field, precise statements for GL_2 over $\mathbb{Q}$ have been proved by Breuil, Colmez, Emerton and Kisin culminating in the proof of many cases of the Fontaine-Mazur conjecture in dimension two (see [E2] and [K]). One of the main tools in these programs is the completed cohomology introduced by Emerton [E1] and the associated spectral sequence. Little is known about it in dimension > 1, in particular for GL_2 over a totally real number field. Our theorem 3.8 implies the degeneracy of the simplest yet non-trivial piece of Emerton's spectral sequence for GL_2 over F, which is its nearly ordinary part (on the Galois side, nearly ordinary means locally reducible at p).

0.3. ***p*-adic *L*-functions in families.** In this work, we call Hida family a local component of the universal Hecke algebra. It turns out that under certain assumptions on the residual representation $\rho_{\pi,p} \mod p$ such components can be identified with universal deformation rings. As explained in [EPW], in the case $F = \mathbb{Q}$, the original definition of a Hida family would then correspond to a branch of our family.

These families are parametrized by irreducible representations

$$\bar{\rho} : \mathrm{Gal}_{F,p\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p),$$

which are nearly ordinary at all places dividing p and totally odd at infinity, where $\mathrm{Gal}_{F,p\Sigma}$ denotes the Galois group of the maximal extension of F in $\overline{\mathbb{Q}}$ unramified outside $p\Sigma\infty$.

Consider the following assumptions on $\bar{\rho}$:

($\star$) no twist of $\bar{\rho}$ extends to a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ for any strict subfield F' of F and its image contains $\mathrm{SL}_2(\mathbb{F}_p)$.

($\star\star$) p is unramified in F and $\bar{\rho} \simeq \rho_{\bar{\pi},p} \mod p$ for some cuspidal automorphic representation $\bar{\pi}$ of GL_2/F which is nearly ordinary and unramified at all places dividing p and has cohomological weight $(\bar{w}, \bar{w}_0)$ such that $\bar{w}_\tau > 0$ for all $\tau \in I$, $\bar{w}_0 = \max_{\tau\in I}(\bar{w}_\tau)$ and $p - 1 > \sum_{\tau\in I}(\frac{\bar{w}_0+\bar{w}_\tau}{2} + 1)$.

Note that, if a cuspidal automorphic representation π of GL_2/F is neither a theta series, nor a twist of a base change, then $\rho_{\pi,p} \mod p$ satisfies ($\star$) for all but finitely many primes p.

The following theorems prove the existence of analytic p-adic L-functions for (nearly) ordinary families of Hilbert automorphic forms, extending results of Kitagawa [Ki], Greenberg-Stevens [GS] and Emerton-Pollack-Weston [EPW] for $\mathbb{Q}$. A novelty of our approach (even for $F = \mathbb{Q}$) is that the p-adic L-functions are naturally elements of universal deformation rings of Galois representations, rather than abstract Iwasawa algebras, which confirms an expectation of Greenberg [Gr, §4].

Let $\mathcal{R}^{\mathrm{n.o}}_{\bar{\rho},\Sigma}$ be the universal $\mathcal{O}[[\mathrm{Gal}(F^{(p\Sigma)}/F) \times (\mathfrak{o}\otimes\mathbb{Z}_p)^{\times p-\mathrm{part}}]]$-algebra parametrizing nearly ordinary deformations of $\bar{\rho}$ (see §4.1) and let $\mathcal{R}^{\det}_{\bar{\rho},\Sigma}$ be the $\mathcal{O}[[(\mathfrak{o}\otimes\mathbb{Z}_p)^{\times p-\mathrm{part}}]]$-algebra parametrizing those deformations having determinant $\det(\rho_{\bar{\pi},p})$ (see (20)).

Theorem 0.2. *Under the assumptions ($\star$) and ($\star\star$), there exists a p-adic L-function*

$$L_p^{\mathrm{n.o}}(\bar{\rho}, \Sigma) \in \mathcal{R}^{\mathrm{n.o}}_{\bar{\rho},\Sigma} = \mathcal{R}^{\det}_{\bar{\rho},\Sigma}[[\mathrm{Gal}(F^{(p\Sigma)}/F)]],$$

unique *up to an element in* $\mathcal{R}_{\bar{\rho},\Sigma}^{\det,\times}$, *whose specialization by any homomorphism* $\mathcal{R}_{\bar{\rho},\Sigma}^{\det} \to \mathcal{O}$ *whose restriction to* $(\mathfrak{o}\otimes\mathbb{Z}_p)^{\times p-\mathrm{part}}$ *is a finite order character, yields the* p*-adic* L*-function* $L_p^{\Sigma}(\pi)$ *of a nearly ordinary cuspidal automorphic representation* π *on* GL_2/F *of parallel weight* $\bar{w}_0$.

For ordinary $\bar{\rho}$, let $\mathcal{R}_{\bar{\rho},\Sigma}^{\mathrm{ord}}$ be the universal ordinary deformation $\mathcal{O}[[\mathrm{Gal}(F^{(p\Sigma)}/F)]]$-algebra (see (19)). A homomorphism $\mathcal{R}_{\bar{\rho},\Sigma}^{\mathrm{ord}} \to \mathcal{O}$ is *algebraic* if its restriction to $\mathrm{Gal}(F^{(p\Sigma)}/F)$ is the product of a finite order character with a non-positive integer power of $\chi\omega^{-1}$.

Theorem 0.3. *Assume that* $\bar{\rho}$ *satisfies* $(\star)$ *and* $(\star\star)$ *with an ordinary* $\bar{\pi}$ *of parallel weight. Then there exists a* p*-adic* L*-function* $L_p^{\mathrm{ord}}(\bar{\rho},\Sigma) \in \mathcal{R}_{\bar{\rho},\Sigma}^{\mathrm{ord}}[[\mathrm{Gal}(F^{(p\Sigma)}/F)]]$ *uniquely determined, up to an element in* $\mathcal{R}_{\bar{\rho},\Sigma}^{\mathrm{ord},\times}$, *by the following universal property: the specialization of* $L_p^{\mathrm{ord}}(\bar{\rho},\Sigma)$ *by* any *algebraic homomorphism* $\mathcal{R}_{\bar{\rho},\Sigma}^{\mathrm{ord}} \to \mathcal{O}$ *yields the* p*-adic* L*-function* $L_p^{\Sigma}(\pi)$ *of a parallel weight, ordinary, cuspidal automorphic representation* π *of* GL_2/F.

The proof of theorem 0.3 (resp. theorem 0.2) relies on the fact that $\mathcal{R}_{\bar{\rho},\Sigma}^{\mathrm{ord}}$ (resp. $\mathcal{R}_{\bar{\rho},\Sigma}^{\mathrm{n.o}}$) is canonically isomorphic to Hida's universal (nearly) ordinary Hecke algebra and that certain modules of (nearly) ordinary cohomology of Hilbert modular varieties are free over these rings (see theorem 4.6 and corollaries 4.8, 4.9).

We believe that using some ideas of Ash-Stevens [AS] and Urban [Ur] would allow to extend our results to the finite slope case, or at least to relax the assumptions $(\star)$ and $(\star\star)$ in the nearly ordinary case as in [GS] (see also [BL]).

Finally we hope that our p-adic L-functions will be useful for the study of higher order partial derivatives as in [M2], and for a formulation and proof of Iwasawa Main Conjectures for GL_2 over totally real fields.

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1. Automorphic symbols on Hilbert Modular Varieties

1.1. **Hilbert modular varieties.** The ring $\mathbb{A}$ of adeles of F is the product of finite adeles $\mathbb{A}_f = F \otimes \widehat{\mathbb{Z}}$ and infinite adeles $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. We denote by F_∞^+ the connected component of 1 in $F_\infty^\times$.

Definition 1.1. For an open compact subgroup K of $\mathrm{GL}_2(\mathbb{A}_f)$ we define the analytic Hilbert modular variety of level K as

$$Y_K = \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})/KK_\infty^+,$$

where K_∞^+ is the subgroup of $\mathrm{GL}_2(F_\infty)$ generated by its center $F_\infty^\times$ and by the connected component of identity in the standard maximal compact subgroup.

The minimal (or Baily-Borel-Satake) compactification $\overline{Y}_K$ of Y_K is obtained by adding finitely many points (the cusps). A basis of neighborhoods for the cusp at infinity is given by the sets

$$\left\{ \mathrm{GL}_2(F) \begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix} KK_\infty^+ \middle| y \in F_\infty^+, \, \mathrm{N}_{F/\mathbb{Q}}(y_\infty) > H \right\}, \text{ for } H \in \mathbb{R}_+^\times, \tag{1}$$

and neighborhoods of other cusp are obtained by translating those by the group action.

The adjoint Hilbert modular variety of level K is defined as

$$Y_K^{\mathrm{ad}} = \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})/ \, \mathbb{A}^\times \, KK_\infty^+.$$

We will only consider K factoring as a product $\prod_v K_v$ over the finite places v of F and such that Y_K^{ad} is smooth. Then Y_K is a finite abelian cover of Y_K^{ad} with group the class group $\mathbb{A}_f^\times / F^\times (\mathbb{A}_f^\times \cap K)$.

Since our interest is in p-adic aspects, usually K_v will be fixed for v not dividing p and for $\alpha \geq 1$, $Y_0(p^\alpha)$, $Y_1(p^\alpha)$, $Y_{11}(p^\alpha)$ and $Y(p^\alpha)$ will denote, respectively, the Hilbert modular varieties whose level $K_p = \prod_{v|p} K_v$ at p equals:

$$\begin{aligned}
K_0(p^\alpha) &= \left\{ u \in \mathrm{GL}_2(\mathfrak{o} \otimes \mathbb{Z}_p) | u \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p^\alpha} \right\}, \\
K_1(p^\alpha) &= \left\{ u \in \mathrm{GL}_2(\mathfrak{o} \otimes \mathbb{Z}_p) | u \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{p^\alpha} \right\}, \\
K_{11}(p^\alpha) &= \left\{ u \in \mathrm{GL}_2(\mathfrak{o} \otimes \mathbb{Z}_p) | u \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{p^\alpha} \right\}, \\
K(p^\alpha) &= \left\{ u \in \mathrm{GL}_2(\mathfrak{o} \otimes \mathbb{Z}_p) | u \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p^\alpha} \right\}.
\end{aligned} \tag{2}$$

Remark 1.2. Classically Hilbert modular varieties are defined as quotients of the d-fold product of upper half planes by congruence subgroups of $\mathrm{SL}_2(\mathfrak{o})$. These occur as connected components of Y_K, but are not preserved by Hecke correspondences, hence the importance of the above adelic definition.

1.2. **Automorphic symbols in level K_1.** The automorphic symbols considered in this section will be sufficient for the construction of the p-adic L-function of a nearly ordinary Hilbert automorphic newform (see theorem 2.5). The definition of automorphic symbols in arbitrary level is postponed to §1.3 and can be skipped at first reading.

1.2.1. *Automorphic cycles and the mirabolic group.* The mirabolic group M is defined as the semi-direct product $\mathbb{G}_m \ltimes \mathbb{G}_a$, where $\mathbb{G}_m$ acts on $\mathbb{G}_a$ by multiplication. A natural embedding of M in GL_2 is given by $(y,x) \mapsto \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$.

Fix an integral ideal $\mathfrak{a}$ of F and denote by a_v the valuation of $\mathfrak{a}$ at a finite place v. Put $M(\mathfrak{a}) = U(\mathfrak{a}) \ltimes (\mathfrak{o} \otimes \widehat{\mathbb{Z}})$, where $U(\mathfrak{a})$ is the open compact subgroup of $(\mathfrak{o} \otimes \widehat{\mathbb{Z}})^\times$ whose elements are congruent to 1 modulo $\mathfrak{a}$. For all v, choose an uniformizer ϖ_v of F_v.

The map:

$$C(\mathfrak{a}) : \mathbb{A}^\times / F^\times U(\mathfrak{a}) \longrightarrow M(F)\backslash M(\mathbb{A})/M(\mathfrak{a}) \ , \ y \mapsto \big(y, (y_v \varpi_v^{-a_v})_{v|\,\mathfrak{a}}\big) \tag{3}$$

is well defined, since for all $z \in F^\times$ and $u \in U(\mathfrak{a})$ we have

$$\big(zyu, ((zyu)_v \varpi_v^{-a_v})_{v|\,\mathfrak{a}}\big) = (z,0)\,\big(y, (y_v \varpi_v^{-a_v})_{v|\,\mathfrak{a}}\big)\,\big(u, ((u_v - 1)\varpi_v^{-a_v})_{v|\,\mathfrak{a}}\big)$$

and $(z,0) \in M(F)$ whereas $\big(u, ((u_v - 1)\varpi_v^{-a_v})_{v|\,\mathfrak{a}}\big) \in M(\mathfrak{a})$.

Denote by $E(\mathfrak{a})$ the group of totally positive units in $\mathfrak{o}$, congruent to 1 modulo $\mathfrak{a}$, and by $\mathrm{Cl}_F^+(\mathfrak{a})$ the narrow ray class group $\mathbb{A}^\times / F^\times U(\mathfrak{a}) F_\infty^+$.

Definition 1.3. For $\eta \in \mathrm{Cl}_F^+(\mathfrak{a})$ we define $C(\eta)$ as the restriction of $C(\mathfrak{a})$ to the inverse image $\widetilde{\eta}$ of η using the short exact sequence

$$1 \to F_\infty^+/E(\mathfrak{a}) \to \mathbb{A}^\times / F^\times U(\mathfrak{a}) \to \mathrm{Cl}_F^+(\mathfrak{a}) \to 1. \tag{4}$$

Definition 1.4. Let $\eta \in \mathrm{Cl}_F^+(\mathfrak{a})$ and let $K \subset \mathrm{GL}_2(\mathbb{A}_f)$ be an open compact subgroup containing the image of $M(\mathfrak{a})$ by the natural inclusion of M in GL_2. The *automorphic cycle* $C_K(\eta)$ on the Hilbert modular variety Y_K is defined as the composed map of $C(\eta)$ with the map coming from the natural inclusion $M \subset \mathrm{GL}_2$:

$$\iota_K : M(F)\backslash M(\mathbb{A})/M(\mathfrak{a}) \longrightarrow \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})/KK_\infty^+ = Y_K.$$

1.2.2. *Automorphic symbols for Hilbert modular varieties.* As mentioned in the introduction, the natural generalization in the Hilbert modular case of the geodesic between the cusps 0 and ∞ in the upper half plane is the Shintani cone $F_\infty^+/E(\mathfrak{a})$. In order to define the corresponding automorphic symbol, it needs to be compactified.

By (4), for any choice of an idele $\xi \in \widetilde{\eta} \cap \mathbb{A}_f^\times$, the map $y_\infty \mapsto y_\infty \xi$ yields an isomorphism $F_\infty^+/E(\mathfrak{a}) \simeq \widetilde{\eta}$, hence there is a continuous map:

$$\widetilde{\eta} \longrightarrow \mathbb{R}_+^\times \ , \ y \mapsto \mathrm{N}_{F/\mathbb{Q}}(y\xi^{-1}). \tag{5}$$

By Dirichlet's unit theorem $F_\infty^+/E(\mathfrak{a}) \simeq (\mathbb{R}\,/\,\mathbb{Z})^{d-1} \times \mathbb{R}_+^\times$.

Definition 1.5. Denote by $\overline{\eta}$ the compactification of $\widetilde{\eta} \xrightarrow{\sim} (\mathbb{R}\,/\,\mathbb{Z})^{d-1} \times \mathbb{R}_+^\times$ obtained by adding two points (zero and infinity).

If $d = 2$ then $\overline{\eta}$ is homeomorphic to a sphere. In general, it is homeomorphic to the suspension of the torus $(\mathbb{R}\,/\,\mathbb{Z})^{d-1}$, hence

$$\mathrm{H}_i(\overline{\eta}) \simeq \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ 0 & \text{for } i = 1, \\ \mathrm{H}_{i-1}((\mathbb{R}\,/\,\mathbb{Z})^{d-1}) & \text{for } 2 \le i \le d. \end{cases}$$

Any ξ induces the same orientation on $\overline{\eta}$, in particular, for any $d \geq 2$ we can canonically identify $\mathrm{H}_d(\overline{\eta}) = \mathrm{H}_d(\overline{\eta}, \{0, \infty\})$ and $\mathbb{Z}$.

Lemma 1.6. *$C_K(\eta)$ extends uniquely to a continuous proper map $\overline{C_K(\eta)} : \overline{\eta} \to \overline{Y}_K$.*

Proof. The uniqueness is clear. For the existence of a continuous map $\overline{C_K(\eta)}$ one has to show that the image by $C_K(\eta)$ of any sequence $(y_n)_{n\geq 1}$ of points in $\widetilde{\eta}$ tending to zero or infinity, converges to a cusp in $\overline{Y}_K$. Suppose first that $\lim_n \mathrm{N}_{F/\mathbb{Q}}(y_n \xi^{-1}) = +\infty$. It follows then from (1) that the sequence $C_K(\eta)(y_n)$ converges to the cusp at infinity (on the connected component of Y_K corresponding to ξ).

Suppose next that $\lim_n \mathrm{N}_{F/\mathbb{Q}}(y_n \xi^{-1}) = 0$ and consider the following diagram:

$$\begin{tikzcd}
(F \cap \xi(\mathfrak{o} \otimes \widehat{\mathbb{Z}}))\backslash F_\infty \arrow[r, "\sim"] \arrow[d] & \mathbb{A}/(F + \xi(\mathfrak{o} \otimes \widehat{\mathbb{Z}})) \arrow[d] & \\
E(\mathfrak{a}) \ltimes (F \cap \xi(\mathfrak{o} \otimes \widehat{\mathbb{Z}}))\backslash M_\infty^+ \arrow[r, hook, "\cdot(\xi,0)"] \arrow[d] & M(F)\backslash M(\mathbb{A})/M(\mathfrak{a}) \arrow[r, two heads] \arrow[d] & M(F)\backslash M(\mathbb{A})/M(\mathfrak{a})M_\infty^+ \arrow[d, equal] \\
E(\mathfrak{a})\backslash F_\infty^+ \arrow[r, hook, "\cdot\xi"] & \mathbb{A}^\times/F^\times U(\mathfrak{a}) \arrow[r, two heads] & \mathrm{Cl}_F^+(\mathfrak{a})
\end{tikzcd}$$

where $M_\infty^+ = F_\infty^+ \ltimes F_\infty$.

Note that for any $y = y_\infty \xi \in \widetilde{\eta}$ the element $y_{\mathfrak{a}}\, \mathfrak{a}^{-1} = (\xi_v \varpi_v^{-a_v})_{v|\,\mathfrak{a}}$ has finite order in

$$\mathbb{A}/(F + \xi(\mathfrak{o} \otimes \widehat{\mathbb{Z}})) \simeq (F \cap \xi(\mathfrak{o} \otimes \widehat{\mathbb{Z}}))\backslash F_\infty,$$

and denote by $x_\infty \in F_\infty$ any element in its coset. Clearly $x_\infty \in F$ and we will show that $C_K(\eta)(y_n)$ converges to the corresponding cusp. In fact, since $\lim_n \mathrm{N}_{F/\mathbb{Q}}(y_n \xi^{-1}) = 0$, the sequence

$$\begin{pmatrix} 0 & 1 \\ -1 & x_\infty \end{pmatrix} \begin{pmatrix} y_n & (\xi_v \varpi_v^{-a_v})_{v|\,\mathfrak{a}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -y_n & 0 \end{pmatrix}$$

converges to cusp at infinity, hence $C_K(\eta)(y_n)$ converges to the cusp determined by x_∞.

Since the map $\overline{C_K(\eta)}$ is finite, to show its properness it is enough to prove that it is closed. This follows easily from the fact that $\overline{C_K(\eta)}$ is continuous and $\overline{\eta}$ compact. $\square$

Definition 1.7. The *automorphic symbol* $S_K(\eta) \in \mathrm{H}_d(\overline{Y}_K)$ is defined as the image of $1 \in \mathbb{Z} \simeq \mathrm{H}_d(\overline{\eta})$ by $\overline{C_K(\eta)}$.

1.2.3. *p-adic automorphic symbols.* Let $\alpha \geq 1$ be an integer such that $K_p \supset K_{11}(p^\alpha)$.

Definition 1.8. For $\beta \geq \alpha$ we define

$$S_{K,\beta} = \sum_{\eta \in \mathrm{Cl}_F^+(p^\beta)} S_K(\eta)[\eta] \in \mathrm{H}_d(\overline{Y}_K, \mathcal{O})[\mathrm{Cl}_F^+(p^\beta)].$$

Remark 1.9. (i) Here one can see one advantage of the adelic approach. The automorphic symbol is defined on the whole class group $\mathrm{Cl}_F^+(p^\beta)$ in contrast with a collection of modular symbols, indexed by $\mathrm{Cl}_F^+(\mathfrak{o})$, each one defined on $(\mathfrak{o}/p^\beta)^\times/E(\mathfrak{o})$.

(ii) Although some of the constructions are similar to those in [Ma] in order to make the relations precise one has to choose representatives in class groups, which would bring some cumbersome notations.

The compatibility of $S_{K,\beta}$ with respect to β, known as distribution property, is governed by the Hecke operator $U_p = \prod_{v|p} U_v^{e_v}$, where e_v denotes the inertia degree at v, so that $(p) = \prod_{v|p} v^{e_v}$. Assume for the rest of this section that for all v dividing p we have $\varpi_v^{e_v} = p$.

Lemma 1.10. *For $\beta \geq \alpha$, the image of $S_{K,\beta+1}$ by the natural projection induced by $\mathrm{pr} : \mathrm{Cl}_F^+(p^{\beta+1}) \to \mathrm{Cl}_F^+(p^\beta)$ equals $U_p \cdot S_{K,\beta}$.*

Proof. It is enough to show that for all $\eta \in \mathrm{Cl}_F^+(p^\beta)$, $U_p \cdot C_K(\eta)$ and $\coprod_{\mathrm{pr}(\eta')=\eta} C_K(\eta')$ define the same cycles on Y_K. We will check this by a computation in the mirabolic group $M(F)\backslash M(\mathbb{A})/M(p^\alpha)$ on which U_p acts according to the following double coset decomposition:

$$M(p^\alpha)(p,0)M(p^\alpha) = \coprod_{b \pmod p} (p,b)M(p^\alpha).$$

For $y \in \mathbb{A}$ we denote by y_p the adele $(y_v)_{v|p}$. We have

$$\begin{aligned}
&U_p \cdot \Big(y \in \mathbb{A}^\times/F^\times U(p^\beta) \mapsto M(F)(y, y_p p^{-\beta})M(p^\alpha)\Big) = \\
&= \coprod_{b \pmod p} \Big(y \in \mathbb{A}^\times/F^\times U(p^\beta) \mapsto M(F)(yp, y_p(1+bp^\beta)p^{-\beta})M(p^\alpha)\Big) = \\
&= \coprod_{b \pmod p} \Big(y \in \mathbb{A}^\times/F^\times U(p^\beta) \mapsto M(F)(y(1+bp^\beta), y_p(1+bp^\beta)p^{-(\beta+1)})M(p^\alpha)\Big) = \\
&= \Big(y \in \mathbb{A}^\times/F^\times U(p^{\beta+1}) \mapsto M(F)(y, y_p p^{-(\beta+1)})M(p^\alpha)\Big).
\end{aligned}$$

□

Remark 1.11. When $F = \mathbb{Q}$ the image by U_p of the geodesic from $x \in \mathbb{Q}$ to ∞ is a union of p geodesics each joining $\frac{x+b}{p}$ to ∞ $(b = 0, ..., p-1)$. When $F \neq \mathbb{Q}$ the situation is quite different. For example if $d = 2$, for β large enough the image by U_p of the 2-cycle $F_\infty^+/E(p^\beta)$ is a union of p^2 2-chains piecing together into $[\mathrm{Cl}_F^+(p^{\beta+1}) : \mathrm{Cl}_F^+(p^\beta)] = p$ copies of the 2-cycle $F_\infty^+/E(p^{\beta+1})$. Note that, in some sense, applying U_p reduces the monodromy of the units.

By lemma 1.10 we can define the *p-adic automorphic symbol* of level K:

$$S_K = \varprojlim_\beta U_p^{-\beta} \cdot S_{K,\beta} \in e_p^* \mathrm{H}_d(\overline{Y}_K, \mathcal{O})[[\mathrm{Cl}_F^+(p^\infty)]],$$

where e_p^* denotes Hida's ordinary idempotent on the homology.

It is clear that the natural projection induced by $\overline{Y_1(p^{\alpha+1})} \to \overline{Y_1(p^\alpha)}$ sends $S_{K_1(p^{\alpha+1})}$ to $S_{K_1(p^\alpha)}$. The universal ordinary p-adic automorphic symbol is defined as:

$$\varprojlim_\alpha S_{K_1(p^\alpha)} \in \varprojlim_\alpha e_p^* \mathrm{H}_d(\overline{Y_1(p^\alpha)}, \mathcal{O}) \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\mathrm{Cl}_F^+(p^\infty)]].$$

One can construct similarly automorphic symbols on adjoint Hilbert modular varieties and define the universal nearly ordinary p-adic automorphic symbol as:

$$\varprojlim_{\alpha} S^{\mathrm{ad}}_{K_{11}(p^\alpha)} \in \varprojlim_{\alpha} e_p^* \, \mathrm{H}_d(\overline{Y^{\mathrm{ad}}_{11}(p^\alpha)}, \mathcal{O}) \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\mathrm{Cl}_F^+(p^\infty)]].$$

1.3. **Automorphic symbols in arbitrary level.**

1.3.1. *Automorphic cycles.* We keep the notations from the previous section. In particular we denote by a_v the valuation of $\mathfrak{a}$ at v. For any ideal $\mathfrak{a}' \subset \mathfrak{a}$ we let $M(\mathfrak{a}, \mathfrak{a}') = U(\mathfrak{a}') \ltimes (\mathfrak{a}' \, \mathfrak{a}^{-1} \otimes \widehat{\mathbb{Z}})$. As in (3), the following map is well defined:

$$C(\mathfrak{a}, \mathfrak{a}') : \mathbb{A}^\times / F^\times U(\mathfrak{a}') \longrightarrow M(F) \backslash M(\mathbb{A}) / M(\mathfrak{a}, \mathfrak{a}') \ , \ y \mapsto \big(y, (y_v \varpi_v^{-a_v})_{v | \, \mathfrak{a}}\big) \, . \tag{6}$$

For any ideal $\mathfrak{a}'' \subset \mathfrak{a}'$, the following diagram is commutative, where the vertical maps are the natural projections:

$$\begin{array}{ccc} \mathbb{A}^\times / F^\times U(\mathfrak{a}'') & \xrightarrow{\quad C(\mathfrak{a}, \mathfrak{a}'') \quad} & M(F) \backslash M(\mathbb{A}) / M(\mathfrak{a}, \mathfrak{a}'') \\ \downarrow & & \downarrow \\ \mathbb{A}^\times / F^\times U(\mathfrak{a}') & \xrightarrow{\quad C(\mathfrak{a}, \mathfrak{a}') \quad} & M(F) \backslash M(\mathbb{A}) / M(\mathfrak{a}, \mathfrak{a}'). \end{array}$$

For $\eta' \in \mathrm{Cl}_F^+(\mathfrak{a}')$ we denote by $C(\mathfrak{a}, \eta')$ the restriction of $C(\mathfrak{a}, \mathfrak{a}')$ to the inverse image $\widetilde{\eta'}$ of η' using the short exact sequence (4).

For a prime w dividing $\mathfrak{a}$, the following lemma describes the action of U_w on an automorphic cycle of level $\mathfrak{a}$.

Lemma 1.12. *For all $\eta \in \mathrm{Cl}_F^+(\mathfrak{a}')$ the cycles $U_w \cdot C(\mathfrak{a}, \eta)$ and $\coprod_{\mathrm{pr}(\varpi_w^{-1} \eta') = \eta} C(\mathfrak{a} \, w, \eta')$ on $M(F) \backslash M(\mathbb{A}) / M(\mathfrak{a}, \mathfrak{a}')$ are cohomologically equivalent, where $\mathrm{pr} : \mathrm{Cl}_F^+(\mathfrak{a}' \, w) \to \mathrm{Cl}_F^+(\mathfrak{a}')$ denotes the natural projection.*

Proof. Let $a_0 = \mathrm{val}_w(\mathfrak{a}' \, \mathfrak{a}^{-1})$. For v dividing $\mathfrak{a}$ put $a'_v = a_v + 1$, if $v = w$, and $a'_v = a_v$, otherwise. We have the following double coset decomposition:

$$M(\mathfrak{a}, \mathfrak{a}') (\varpi_w, 0) M(\mathfrak{a}, \mathfrak{a}') = \coprod_{b \pmod{w}} (\varpi_w, b \varpi_w^{a_0}) M(\mathfrak{a}, \mathfrak{a}').$$

Hence $U_w \cdot C(\mathfrak{a}, \mathfrak{a}') = U_w \cdot \big(y \in \mathbb{A}^\times / F^\times U(\mathfrak{a}') \mapsto M(F) \left(y, (y_v \varpi_v^{-a_v})_{v | \mathfrak{a}}\right) M(\mathfrak{a}, \mathfrak{a}')\big) =$

$$= \coprod_{b \pmod{w}} \big(y \in \mathbb{A}^\times / F^\times U(\mathfrak{a}') \mapsto M(F) \left(y \varpi_w, \xi_{w,b} (y_v \varpi_v^{-a_v})_{v | \mathfrak{a}}\right) M(\mathfrak{a}, \mathfrak{a}')\big) \, ,$$

where $\xi_{w,b}$ is the idele equal to $1 + b \varpi_w^{a_0 + a_w}$ at w and 1 at all other places. Since $a_0 + a_w = \mathrm{val}_w(\mathfrak{a}')$, we have $\xi_{w,b} \in U(\mathfrak{a}')$, hence $U_w \cdot C(\mathfrak{a}, \mathfrak{a}')$ equals:

$$\coprod_{b \pmod{w}} \big(y \in \mathbb{A}^\times / F^\times U(\mathfrak{a}') \mapsto M(F) \left(\xi_{w,b} y \varpi_w, \xi_{w,b} (y_v \varpi_v^{-a_v})_{v | \mathfrak{a}}\right) M(\mathfrak{a}, \mathfrak{a}')\big) \, .$$

By the change of variable $y \mapsto y\varpi_w^{-1}$ this is homologically equivalent to

$$\coprod_{b \pmod{w}} \left(y \in \mathbb{A}^\times / F^\times U(\mathfrak{a}') \mapsto M(F)\left(\xi_{w,b} y, (\xi_{w,b} y_v \varpi_v^{-a'_v})_{v|\mathfrak{a}}\right) M(\mathfrak{a},\mathfrak{a}')\right) =$$

$$= \left(y \in \mathbb{A}^\times / F^\times U(\mathfrak{a}' \, w) \mapsto M(F)\left(y, (y_v \varpi_v^{-a'_v})_{v|\mathfrak{a}}\right) M(\mathfrak{a},\mathfrak{a}')\right),$$

which is the composition of $C(w\,\mathfrak{a}, w\,\mathfrak{a}')$ with the natural projection:

$$M(F)\backslash M(\mathbb{A})/M(w\,\mathfrak{a}, w\,\mathfrak{a}') \longrightarrow M(F)\backslash M(\mathbb{A})/M(\mathfrak{a},\mathfrak{a}').$$

□

1.3.2. *Automorphic symbols for Hilbert modular varieties.* Let K be any open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$. Given integral ideals $\mathfrak{a}' \subset \mathfrak{a}$ such that K contains the image of $M(\mathfrak{a},\mathfrak{a}') = U(\mathfrak{a}') \ltimes (\mathfrak{a}'\,\mathfrak{a}^{-1} \otimes \widehat{\mathbb{Z}})$ in GL_2, we denote by $C_K(\mathfrak{a},\mathfrak{a}')$ the composition of $C(\mathfrak{a},\mathfrak{a}')$ with the following map coming from the natural inclusion $M \subset \mathrm{GL}_2$:

$$\iota_K : M(F)\backslash M(\mathbb{A})/M(\mathfrak{a},\mathfrak{a}') \longrightarrow \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})/KK_\infty^+ = Y_K.$$

Definition 1.13. Let $\mathfrak{a}_K$ be an integral ideal of F, such that for any $\mathfrak{a}$, K contains the image of $M(\mathfrak{a},\mathfrak{a}\,\mathfrak{a}_K)$ in GL_2, so that the map $C_K(\mathfrak{a},\mathfrak{a}\,\mathfrak{a}_K)$ is well defined.

For $\eta' \in \mathrm{Cl}_F^+(\mathfrak{a}')$ let $C_K(\mathfrak{a},\eta') = \iota_K \circ C(\mathfrak{a},\eta')$ be the restriction of $C_K(\mathfrak{a},\mathfrak{a}')$ to $\widetilde{\eta'}$.

As in lemma 1.6 one can prove the existence and uniqueness of a proper continuous map $\overline{C_K(\mathfrak{a},\eta')} : \overline{\eta'} \to \overline{Y}_K$ extending $C_K(\mathfrak{a},\eta')$ and define $S_K(\mathfrak{a},\eta') \in \mathrm{H}_d(\overline{Y}_K)$ as the image of $1 \in \mathrm{H}_d(\overline{\eta'})$ by $\overline{C_K(\mathfrak{a},\eta')}$.

1.3.3. *p-adic automorphic symbols.* Since $\mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma}$ is naturally a $\mathcal{O}[[\mathrm{Gal}(F^{(p\Sigma)}/F)]]$-algebra, it will be useful to define automorphic symbols indexed by $\mathrm{Cl}_F^+(p^\infty\Sigma)$ and not only by $\mathrm{Cl}_F^+(p^\infty)$ as in §1.2.3. For this, we fix a finite set Σ of auxiliary primes not dividing p and let Σ also denote their product.

Let $K \subset \mathrm{GL}_2(\mathbb{A}_f)$ be an open compact subgroup and fix an ideal $\mathfrak{a}_K$ as in definition 1.13.

Definition 1.14. Given an integer $\beta \geq 1$, we define:

$$S_{K,\beta}^\Sigma = \sum_{\eta \in \mathrm{Cl}_F^+(p^\beta\Sigma\,\mathfrak{a}_K)} S_K(p^\beta\Sigma,\eta)[\eta\Sigma^{-1}\prod_{v|p}\varpi_v^{-e_v\beta}] \in \mathrm{H}_d(\overline{Y}_K,\mathcal{O})[\mathrm{Cl}_F^+(p^\beta\Sigma\,\mathfrak{a}_K)]$$

Lemma 1.12 has the following consequence:

Corollary 1.15. *The image of $S_{K,\beta+1}^\Sigma$ by the natural projection induced by* pr : $\mathrm{Cl}_F^+(p^{\beta+1}\Sigma\,\mathfrak{a}_K) \to \mathrm{Cl}_F^+(p^\beta\Sigma\,\mathfrak{a}_K)$ *equals $U_p \cdot S_{K,\beta}^\Sigma$.*

By corollary 1.15 we can define the *p-adic automorphic symbol* of level K:

$$S_K^\Sigma = \varprojlim_\beta U_p^{-\beta} \cdot S_{K,\beta}^\Sigma \in e_p^* \,\mathrm{H}_d(\overline{Y}_K,\mathcal{O})[[\mathrm{Cl}_F^+(p^\infty\Sigma\,\mathfrak{a}_K)]], \tag{7}$$

where e_p^* denotes Hida's ordinary idempotent on the homology.

1.4. **Cohomology of Hilbert modular varieties.**

1.4.1. *Cohomological weights.* The characters of the torus $\mathrm{Res}^F_{\mathbb{Q}}\,\mathbb{G}_\mathrm{m}$ can be identified with $\mathbb{Z}[I]$ as follows: for any $w=\sum_{\tau\in I} w_\tau\tau\in\mathbb{Z}[I]$ and for any $\mathbb{Q}$-algebra A splitting $F^\times$, we consider the character $x\in(F\otimes_{\mathbb{Q}}A)^\times\mapsto x^w=\prod_{\tau\in I}\tau(x)^{w_\tau}\in A^\times$. The norm character $\mathrm{N}_{F/\mathbb{Q}}:\mathrm{Res}^F_{\mathbb{Q}}\,\mathbb{G}_\mathrm{m}\to\mathbb{G}_\mathrm{m}$ then corresponds to the element $t=\sum_{\tau\in I}\tau\in\mathbb{Z}[I]$.

Definition 1.16. (i) A weight $(w,w_0)\in\mathbb{Z}[I]\times\mathbb{Z}$ is *cohomological* if for all $\tau\in I$ we have $w_\tau\geq 0$ and $w_\tau\equiv w_0\pmod 2$.
(ii) A cohomological weight (w,w_0) is *critical* if $|w_0|\leq\min_{\tau\in I}(w_\tau)$. It is *parallel* if $w=w_0t$.

Remark 1.17. The correspondence with the classical notion of weight of a Hilbert modular form is as follows. Let f be a Hilbert modular newform of weight $(k_\tau)_{\tau\in I}$, $k_\tau\geq 2$ of the same parity, generating a cuspidal automorphic representation π. According to [Da] and [Pa], the value $L(f,m)$ is critical, if for all τ we have $\frac{k_0-k_\tau}{2}+1\leq m\leq\frac{k_0+k_\tau}{2}-1$, where $k_0=\max_\tau(k_\tau)$. Put $w_0=k_0-2m$ and $w_\tau=k_\tau-2$ for $\tau\in I$. Then $L(f,m)=L(\pi\otimes|\cdot|_{\mathbb{A}}^{\frac{k_0-w_0}{2}-1},1)$ where the automorphic representation $\pi\otimes|\cdot|_{\mathbb{A}}^{\frac{k_0-w_0}{2}-1}$ has cohomological weight (w,w_0) which is critical (see definition 2.1).

An irreducible algebraic representation of $\mathrm{Res}^F_{\mathbb{Q}}\,\mathrm{GL}_2$ with central action factoring through the norm is necessarily of the form

$$\bigotimes_{\tau\in I}\mathrm{Sym}_\tau^{w_\tau}\otimes\mathrm{det}_\tau^{(w_0-w_\tau)/2},$$

for some cohomological weight $(w,w_0)\in\mathbb{Z}[I]\times\mathbb{Z}$. For an $\mathcal{O}$-module A we denote by $\mathcal{L}(w,w_0;A)$ the corresponding $A[\mathrm{GL}_2(\mathfrak{o}\otimes\mathcal{O})]$-module, where we use the fixed embedding of $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}}_p$ to identify $\mathrm{GL}_2(\mathfrak{o}\otimes\mathcal{O})$ with $\mathrm{GL}_2(\mathcal{O})^I$. It can be realized as the space of polynomials in $(X=(X_\tau)_{\tau\in I},Y=(Y_\tau)_{\tau\in I})$ which are homogeneous of degree w_τ in the variables (X_τ,Y_τ).

1.4.2. *Sheaf cohomology.* Let (w,w_0) be a cohomological weight. Then

$$\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})\times\mathcal{L}(w,w_0;A)/KK_\infty^+,$$

with KK_∞^+ acting on $\mathcal{L}(w,w_0;A)$ on the left via K_p, is a local system on Y_K, and we denote by $\mathcal{L}_K(w,w_0;A)$ the corresponding sheaf of locally constant sections.

For $K'\subset K$, there is a natural projection $\mathrm{pr}:Y_{K'}\to Y_K$ and $\mathrm{pr}^*\mathcal{L}_K(w,w_0;A)=\mathcal{L}_{K'}(w,w_0;A)$. We denote $\mathrm{H}^\bullet(Y_K,\mathcal{L}_K(w,w_0;A))$ the corresponding singular (or Betti) cohomology groups, and by $\mathrm{H}^\bullet_c(Y_K,\mathcal{L}_K(w,w_0;A))$ the cohomology with compact support.

1.4.3. *Hecke operators.* For $g\in\mathrm{GL}_2(\mathbb{A}_f)$ we define the Hecke correspondence $[KgK]$ on Y_K by the usual diagram:

$$\begin{tikzcd} & Y_{K\cap g^{-1}Kg} \arrow[r, "\cdot g"] \arrow[dl, "\mathrm{pr}_1"'] & Y_{gKg^{-1}\cap K} \arrow[dr, "\mathrm{pr}_2"] & \\ Y_K & & & Y_K \end{tikzcd}$$

We define the standard Hecke operators $T_v = [K_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K_v]$, $S_v = [K_v \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K_v]$, for v outside a certain finite set of bad primes, and $U_v = [K_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K_v]$, for the remaining v.

The Hecke correspondences at infinity are $[K_\infty^+ g_\infty K_\infty^+]$, where g_∞ is an element of the group $\left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)^I \subset \mathrm{GL}_2(F_\infty)$.

The Betti cohomology groups $\mathrm{H}^\bullet(Y_K, \mathcal{L}_K(w, w_0; A))$ admit a natural action of all the Hecke correspondences and the induced endomorphisms commute with each other.

Definition 1.18. Let e_∞ denote the idempotent on $\mathrm{H}^\bullet(Y_K, \mathcal{L}_K(w, w_0; A))$ which cuts out the part fixed by all the Hecke correspondences at infinity.

1.5. **Evaluation of cohomology classes on automorphic symbols.** As mentioned in the introduction, when $F \neq \mathbb{Q}$, the evaluation of cohomology classes on modular symbols requires special care.

By definition the cycle $C_{K,\beta}^{\Sigma} = C_K(p^\beta \Sigma, p^\beta \Sigma\, \mathfrak{a}_K)$ yields a homomorphism:

$$C_{K,\beta}^{\Sigma *} : \mathrm{H}_c^d(Y_K, \mathcal{L}_K(w, w_0; \mathcal{O})) \to \mathrm{H}_c^d(\mathbb{A}^\times / F^\times U(p^\beta \Sigma\, \mathfrak{a}_K), C_{K,\beta}^{\Sigma *} \mathcal{L}_K(w, w_0; \mathcal{O})).$$

The first difficulty comes from the fact that the local system $C_K(\mathfrak{a}, \mathfrak{a}')^* \mathcal{L}_K(w, w_0; \mathcal{O})$ on $\mathbb{A}^\times / F^\times U(\mathfrak{a}')$ is not trivial, because of the monodromy action of the units. Explicitely, it is given by

$$F^\times \backslash \left(\mathbb{A}^\times \times \mathcal{L}(w, w_0; \mathcal{O})\right) / U(\mathfrak{a}'),$$

where $u \in U(\mathfrak{a}')$ acts on $\mathcal{L}(w, w_0; \mathcal{O})$ by $\begin{pmatrix} u_p & ((u_v - 1)\varpi_v^{-a_v})_{v|p} \\ 0 & 1 \end{pmatrix}$ whereas $F^\times$ acts trivially (we recall that u_p denotes the adele $(u_v)_{v|p}$). In particular

$$C_{K,\beta}^{\Sigma *} \mathcal{L}_K(w, w_0; \mathcal{O}) = F^\times \backslash \left(\mathbb{A}^\times \times \mathcal{L}(w, w_0; \mathcal{O})\right) / U(p^\beta \Sigma\, \mathfrak{a}_K),$$

where $u \in U(p^\beta \Sigma\, \mathfrak{a}_K)$ acts on $\mathcal{L}(w, w_0; \mathcal{O})$ by $\begin{pmatrix} u_p & (u_p - 1)p^{-\beta} \\ 0 & 1 \end{pmatrix}$.

Nevertheless, one can extract one coefficient (the critical one) as follows. We first need to untwist this sheaf, an operation which for classical modular symbols is implicit in the definition. The map $(y, v) \mapsto (y, \begin{pmatrix} 1 & -p^{-\beta} \\ 0 & 1 \end{pmatrix} \cdot v)$ induces a homomorphism of sheaves on $\mathbb{A}^\times / F^\times U(p^\beta \Sigma\, \mathfrak{a}_K)$:

$$\mathrm{tw}_\beta^{w, w_0} : C_{K,\beta}^{\Sigma *} \mathcal{L}_K(w, w_0; \mathcal{O}) \longrightarrow F^\times \backslash \left(\mathbb{A}^\times \times \mathcal{L}(w, w_0; E)\right) / U(p^\beta \Sigma\, \mathfrak{a}_K),$$

where $u \in U(p^\beta \Sigma\, \mathfrak{a}_K)$ acts on $\mathcal{L}(w, w_0; E)$ simply by

$$\begin{pmatrix} 1 & -p^{-\beta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_p & (u_p - 1)p^{-\beta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & p^{-\beta} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u_p & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose now that (w, w_0) is critical, so that $X^{(w - w_0 t)/2} Y^{(w + w_0 t)/2} \in \mathcal{L}(w, w_0; \mathcal{O})$.

A direct computation shows that for any $0 \leq j \leq w$, $\left(\begin{smallmatrix} u_p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ acts by $u_p^{(w + w_0 t)/2 - j}$ on $X^{w-j} Y^j \in \mathcal{L}(w, w_0; \mathcal{O})$. In particular,

$$\left(\begin{smallmatrix} u_p & 0 \\ 0 & 1 \end{smallmatrix}\right) \text{ acts trivially on } X^{(w - w_0 t)/2} Y^{(w + w_0 t)/2}. \tag{8}$$

Therefore evaluating at the coefficient in front of $X^{(w-w_0t)/2}Y^{(w+w_0t)/2}$ induces the following homomorphism of sheaves:

$$\mathrm{crit}_\beta^{w,w_0} : F^\times \backslash \left(\mathbb{A}^\times \times \mathcal{L}(w,w_0;E)\right)/U(p^\beta \Sigma\, \mathfrak{a}_K) \longrightarrow \mathbb{A}^\times / F^\times U(p^\beta \Sigma\, \mathfrak{a}_K) \times E.$$

Since $p^{\beta\frac{w-w_0t}{2}}(X-p^{-\beta}Y)^{(w-w_0t)/2}Y^{(w+w_0t)/2} \in \mathcal{L}(w,w_0;\mathcal{O})$, we have the following:

Lemma 1.19. *Assume that (w,w_0) is critical. Then the map $p^{\beta\frac{w-w_0t}{2}}\mathrm{crit}_\beta^{w,w_0}\circ \mathrm{tw}_\beta^{w,w_0}$ takes values in $\mathbb{A}^\times / F^\times U(p^\beta\Sigma\, \mathfrak{a}_K)\times \mathcal{O}$. Further composing with $C_{K,\beta}^{\Sigma *}$ induces*

$$\mathrm{H}_c^d(Y_K, \mathcal{L}_K(w,w_0;\mathcal{O})) \longrightarrow \mathrm{H}_c^d(\mathbb{A}^\times / F^\times U(p^\beta\Sigma\, \mathfrak{a}_K),\mathcal{O}) = \mathcal{O}[\mathrm{Cl}_F^+(p^\beta\Sigma\, \mathfrak{a}_K)]. \tag{9}$$

The second difficulty is related with the action of the Hecke operator U_p on cycles (see remark 1.11).

Consider the natural projection $\mathrm{pr}_{\beta+1}^\beta : \mathrm{Cl}_F^+(p^{\beta+1}\Sigma\, \mathfrak{a}_K) \to \mathrm{Cl}_F^+(p^\beta\Sigma\, \mathfrak{a}_K)$. Lemma 1.12 implies that the cycles $U_p \cdot C_{K,\beta}^\Sigma$ and $C_{K,\beta+1}^\Sigma$ are cohomologically equivalent. This implies the commutativity of the upper square in the following diagram:

(10)

$$\begin{tikzcd}
\mathrm{H}_c^d(Y_K, \mathcal{L}_K(w,w_0;\mathcal{O})) \arrow[r, "p^{\frac{w-w_0t}{2}}U_p = U_p^0"] \arrow[d, "C_{K,\beta+1}^{\Sigma *}"] & \mathrm{H}_c^d(Y_K, \mathcal{L}_K(w,w_0;\mathcal{O})) \arrow[d, "C_{K,\beta}^{\Sigma *}"] \\
\mathrm{H}_c^d(\mathbb{A}^\times / F^\times U(p^{\beta+1}\Sigma\, \mathfrak{a}_K), C_{K,\beta+1}^{\Sigma *}\mathcal{L}_K) \arrow[r, "p^{\frac{w-w_0t}{2}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)"] \arrow[d, "p^{(\beta+1)\frac{w-w_0t}{2}}\mathrm{crit}_{\beta+1}^{w,w_0}\circ \mathrm{tw}_{\beta+1}^{w,w_0}"] & \mathrm{H}_c^d(\mathbb{A}^\times / F^\times U(p^{\beta}\Sigma\, \mathfrak{a}_K), C_{K,\beta}^{\Sigma *}\mathcal{L}_K) \arrow[d, "p^{\beta\frac{w-w_0t}{2}}\mathrm{crit}_{\beta}^{w,w_0}\circ \mathrm{tw}_{\beta}^{w,w_0}"] \\
\mathcal{O}[\mathrm{Cl}_F^+(p^{\beta+1}\Sigma\, \mathfrak{a}_K)] \arrow[r, "\mathrm{pr}_{\beta+1}^\beta"'] & \mathcal{O}[\mathrm{Cl}_F^+(p^{\beta}\Sigma\, \mathfrak{a}_K)]
\end{tikzcd}$$

The commutativity of the lower square follows from the matrix computation:

$$\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & -p^{-\beta-1} \\ 0 & 1\end{pmatrix} = \begin{pmatrix} 1 & p^{-\beta} \\ 0 & 1\end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix},$$

and the fact that $\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)$ acts trivially on $X^{(w-w_0t)/2}Y^{(w+w_0t)/2}$.

The ordinary idempotent e_p acting on the cohomology of quaternionic Shimura varieties (including Hilbert modular varieties) has been introduced by Hida in [H1]. In parallel weight e_p cuts out the maximal direct factor of the p-adic cohomology on which U_p is invertible (i.e. acts by a p-adic unit). In arbitrary cohomological weight (w,w_0) there is a shift by the lowest Hodge-Tate weight of the cohomology which is $(w_0t-w)/2$ and one should consider instead the operator $U_p^0 = p^{\frac{w-w_0t}{2}}U_p$ which preserves the p-adic integral structure in an optimal way. Usually one reserves the word ordinary for level $K_1(p^\alpha)$ and uses the term nearly ordinary otherwise (some authors use the word ordinary for both).

By (10) the maps

$$S_{K,\Sigma,\beta}^{w,w_0} = p^{\beta\frac{w-w_0t}{2}}\mathrm{crit}_\beta^{w,w_0}\circ \mathrm{tw}_\beta^{w,w_0}\circ C_{K,\beta}^{\Sigma *}\circ (U_p^0)^{-\beta}$$

form a projective system, leading to:

Definition 1.20. For critical (w, w_0), we consider the homomorphism:

$$S_{K,\Sigma}^{w,w_0} = \left(S_{K,\Sigma,\beta}^{w,w_0}\right)_{\beta \geq 1} : e_p\,\mathrm{H}_c^d(Y_K, \mathcal{L}_K(w, w_0; \mathcal{O})) \to \mathcal{O}[[\mathrm{Cl}_F^+(p^\infty \Sigma\, \mathfrak{a}_K)]].$$

It will be used in §2.3 to attach p-adic L-functions to Hilbert automorphic forms.

In order to attach p-adic L-functions to newforms (see theorem 2.5), it is enough to take $K = K_{11}(p^\alpha) \cap K_1(\mathfrak{n})$ and consider the homomorphism:

$$S_K^{w,w_0} = \left(S_{K,\varnothing,\beta}^{w,w_0}\right)_{\beta \geq \alpha} : e_p\,\mathrm{H}_c^d(Y_K, \mathcal{L}_K(w, w_0; \mathcal{O})) \to \mathcal{O}[[\mathrm{Cl}_F^+(p^\infty)]]$$

whose construction only relies on the automorphic symbols introduced in §1.2.

2. p-adic L-functions for nearly ordinary automorphic forms on GL_2

We will construct p-adic L-functions for various nearly ordinary Hilbert automorphic forms encoding both the vertical and the horizontal aspects of the theory.

By global class field theory, the maximal abelian pro-p quotient $\mathrm{Gal}(F^{(p\Sigma)}/F)$ of $\mathrm{Gal}_{F,p\Sigma}$ is isomorphic to the maximal pro-p quotient $\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)$ of $\mathrm{Cl}_F^+(p^\infty\Sigma)$ and the character $\chi\omega^{-1}$ can be seen as character of $\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)$.

2.1. **Automorphic representations.**

2.1.1. *Archimedean Euler factors.* Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ and let $(w, w_0) \in \mathbb{Z}[I] \times \mathbb{Z}$ be a cohomological weight (see definition 1.16).

Definition 2.1. We say that π is cohomological of weight (w, w_0) if for all $\tau \in I$ the representation π_τ is parabolically induced from the following character

$$\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \mapsto \mathrm{sgn}(a)^{w_0} |a|^{\frac{1}{2}(w_\tau + 2 - w_0)} |d|^{-\frac{1}{2}(w_\tau + 2 + w_0)}.$$

In what follows we assume that π is cohomological of weight (w, w_0). The definition is justified by the fact that such a π contributes to $\mathrm{H}^d(Y_K, \mathcal{L}_K(w, w_0; \mathbb{C}))$ whenever $\pi_f^K \neq \{0\}$. Moreover, the Harish-Chandra parameter of π_∞ is given by $(w + t, w_0)$ and it is known that such a π corresponds to a Hilbert modular newform of weight $(w + 2t, w_0)$.

One can attach to π a Γ-factor, depending only on π_∞, as follows:

$$\Gamma(\pi, s) = \prod_{\tau \in I} \Gamma_{\mathbb{C}}\left(s - \frac{w_0 - w_\tau}{2}\right), \text{ where } \Gamma_{\mathbb{C}}(s) = 2 \cdot (2\pi)^{-s}\Gamma(s). \tag{11}$$

The functional equation for π relates $L(\pi, s)$ and $L(\check{\pi}, 1 - s) = L(\pi, w_0 + 2 - s)$, hence the value of $L(\pi, s)$ at $s = 1$ is critical in the sense of Deligne if neither $\Gamma(\pi, s)$ nor $\Gamma(\pi, w_0 + 2 - s)$ has a pole at $s = 1$.

It is straightforward to check that $L(\pi, 1)$ is critical in the sense of Deligne [De] if, and only if, (w, w_0) is critical in the sense of definition 1.16, that is $|w_0| \leq \min_{\tau \in I}(w_\tau)$.

2.1.2. *Automorphic ordinariness.* The fixed embeddings of $\overline{\mathbb{Q}}$ in $\mathbb{C}$ and in $\overline{\mathbb{Q}}_p$ yield a partition $I = \coprod_{v|p} I_v$. It is a well known fact [H1, §3] that the Hecke operator $U_v^0 = \left(\prod_{\tau \in I_v} \varpi_v^{e_v \frac{w_\tau - w_0}{2}}\right) U_v$ is a p-adically integral endomorphism of the cohomological automorphic forms in weight (w, w_0).

Definition 2.2. (i) For v dividing p, we say that π_v is ordinary if, either π_v is ramified and the U_v^0-eigenvalue on its new vector is a p-adic unit, or it is unramified and its new vector has a U_v^0-stabilization whose eigenvalue is a p-adic unit.

(ii) For v dividing p, we say that π_v is nearly ordinary if there exists a finite order character $\nu_v : F_v^\times \to \mathbb{C}^\times$ such that $\pi_v \otimes \nu_v^{-1}$ is ordinary. If we further impose $\nu_v(\varpi_v) = 1$, then ν_v is unique, called the ordinary twist type of π_v with respect to ϖ_v.

2.1.3. *Adelic Mellin transform.* Let ϕ be a finite order Hecke character over F. If ϕ is *everywhere unramified* then it is well known (see [Bu]) that $L(\pi \otimes \phi, 1)$ completed by the Euler factors admits the following integral expression:

$$\int_{\mathbb{A}^\times / F^\times} \phi(y) f\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) d^\times y,$$

where f is a certain automorphic form in π and $d^\times y$ denotes the Haar measure on $\mathbb{G}_m$. For arbitrary ϕ, one should consider the non-trivial additive unitary character of $\mathbb{A}/F$:

$$\xi : \mathbb{A}/F \longrightarrow \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \longrightarrow \mathbb{C}^\times,$$

where the first map is the trace, whereas the second is the usual additive character ξ_0 on $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ characterized by $\ker(\xi_0|_{\mathbb{Q}_\ell}) = \mathbb{Z}_\ell$ for every prime number ℓ and $\xi_0|_{\mathbb{R}} = \exp(2i\pi \cdot)$. We have $\ker(\xi_v) = (\varpi_v^{-\delta_v})$, where δ_v is the valuation at v of the different of F.

An automorphic form f admits an adelic Fourier expansion:

$$f(g) = \sum_{y \in F^\times} W\left(\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) g\right), \tag{12}$$

where $W(g) = \int_{\mathbb{A}/F} \overline{\xi}(x) f\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\right) dx$ is the adelic Whittaker function of f with respect to ξ. If f is holomorphic the above sum is in fact supported by $F^\times \cap F_\infty^+$.

Given a ramified character μ_v of $F_v^\times$, the local Gauss sum $\tau_v(\mu_v, \xi_v)$ is given by:

$$\int_{F_v^\times} \mu_v(y) \xi_v(y) d^\times y = \int_{\mathfrak{o}_v^\times} \mu_v\left(\frac{u}{\varpi_v^{\mathrm{cond}(\mu_v) + \delta_v}}\right) \xi_v\left(\frac{u}{\varpi_v^{\mathrm{cond}(\mu_v) + \delta_v}}\right) du. \tag{13}$$

Note that the local Gauss sums do not depend on the choice of the uniformizer ϖ_v.

2.2. **p-adic L-functions attached to newforms.** Let π be a cohomological cuspidal automorphic representation of weight (w, w_0). Assume that for every v dividing p, π_v is nearly ordinary and we let ν_v be as in definition 2.2. For every place v dividing p denote by α_v the eigenvalue of U_v acting on the ordinary U_v-stabilization of a new vector in $\pi_v \otimes \nu_v^{-1}$. Since $\pi_v \otimes \nu_v^{-1}$ is ordinary, α_v is independent of the choice of ϖ_v.

Denote by $\mathfrak{n}$ the prime to p part of the conductor of π.

In addition to the idempotents e_p and e_∞ introduced earlier, let e_π be the idempotent cutting out the π-eigenspace for the action of the Hecke operators T_v and S_v, where v runs over the primes of F outside a sufficiently large finite set of primes (here one can take the primes v not dividing $\mathfrak{n}\, p$).

By the theory of a new vector, for α large enough

$$e_\pi e_\infty e_p \,\mathrm{H}_c^d\left(Y_{K_1(\mathfrak{n})\cap K_{11}(p^\alpha)}, \mathcal{L}_K(w, w_0; \mathbb{C})\right) \tag{14}$$

is a complex line having a basis $\delta_\infty(f_\pi)$, where δ_∞ denotes the Matsushima-Shimura-Harder isomorphism (see [H5] or [D2, §7.1]) and f_π is an automorphic form in π which is nearly ordinary at all places dividing p, new outside p and normalized using its adelic Fourier expansion (12). The complex line (14) contains a natural $\mathcal{O}$-line, namely the torsion-free part of

$$e_\pi e_\infty e_p \,\mathrm{H}_c^d\left(Y_{K_1(\mathfrak{n})\cap K_{11}(p^\alpha)}, \mathcal{L}_K(w, w_0; \mathcal{O})\right),$$

and we fix a basis b_π of the latter. An direct computation shows that U_v acts on b_π by $\alpha_v \nu_v(\varpi_v) = \alpha_v$, if ν_v is the ordinary twist type for π_v with respect to ϖ_v.

Definition 2.3. The Archimedean period $\Omega_{\pi,\infty} = \delta_\infty(f_\pi)/b_\pi \in \mathbb{C}^\times$ is well defined up to an element of $\mathcal{O}^\times$.

Definition 2.4. Assume that the weight (w, w_0) is critical. The primitive (or new) p-adic L-function $L_p(\pi) \in \mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty)]]$ attached to π, is defined as the image of $S^{w,w_0}_{K_1(\mathfrak{n})\cap K_{11}(p^\alpha)}(b_\pi) \in \mathcal{O}[[\mathrm{Cl}_F^+(p^\infty)]]$ by the natural projection (see definition 1.20).

Theorem 2.5. (i) *Let $\phi : \mathrm{Cl}_F^+(p^\infty) \to \mathcal{O}^\times$ be a p-power order character. Then the image of $L_p(\pi)$ by the resulting homomorphism $\mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty)]] \to \mathcal{O}$ equals:*

$$\frac{L^{(p)}(\pi\otimes\phi, 1)\Gamma(\pi, 1)}{\Omega_{\pi,\infty}} \prod_{v|p} Z_v, \text{ where for } v \text{ dividing } p \text{ we have}$$

$$Z_v = \begin{cases} \alpha_v^{-\operatorname{cond}(\phi_v\nu_v)} \tau_v(\phi_v\nu_v, \xi_v), \text{ if } \phi_v\nu_v \text{ is ramified, and} \\ \phi_v(\varpi_v)^{-\delta_v}\left(1 - (\alpha_v\phi_v(\varpi_v)\,\mathrm{N}_{F/\mathbb{Q}}(v))^{-1}\right)(1-\alpha_v\phi_v(\varpi_v))^{-1}, \text{ otherwise.}\end{cases}$$

(ii) *If (w, w_0-2) is critical too, the automorphism $[a] \mapsto (\chi\omega^{-1})(a)[a]$ of $\mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty)]]$ sends $L_p(\pi)$ to $L_p(\pi \otimes |\cdot|_{\mathbb{A}}\omega^{-1})$.*

Remark 2.6. (i) The first part of the theorem uniquely determines $L_p(\pi)$, without assuming Leopoldt's conjecture.

(ii) If π is ordinary at all places v dividing p (that is $\nu_v = 1$), the interpolation formula has a particularly simple form since $\prod_{v|p} \tau_v(\phi_v, \xi_v) = \tau(\phi, \xi)$ is a global Gauss sum.

(iii) It follows from the interpolation formula that $L_p(\pi)$ does not depend upon the choice of uniformizers ϖ_v, for $v \mid p$.

Proof. (i) Fix $\beta \geq \alpha$ so that ϕ can be seen as a character of $\mathrm{Cl}_F^+(p^\beta)$. By (8), $\left(\begin{smallmatrix} y_p & y_p p^{-\beta} \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -p^{-\beta} \\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix} y_p & 0 \\ 0 & 1\end{smallmatrix}\right)$ acts trivially on $X^{(w-w_0t)/2}Y^{(w+w_0t)/2}$.

By unwinding the definition one sees that the specialization of $L_p(\pi)$ by ϕ equals

$$\frac{\prod_{v|p}((\phi_v\nu_v)(\varpi_v)\alpha_v)^{-e_v\beta}}{\Omega_{\pi,\infty}}\int_{\mathbb{A}^\times/F^\times}\phi(y)f_{\pi,1}\left(\begin{smallmatrix} y & y_p p^{-\beta} \\ 0 & 1\end{smallmatrix}\right)d^\times y, \tag{15}$$

where for a character $\epsilon : \{\pm 1\}^I \to \{\pm 1\}$ we put $f_{\pi,\epsilon}(g) = \sum_{J\subset I}\epsilon(J)f_\pi\left(g\left(\begin{smallmatrix} -1_J & 0 \\ 0 & 1\end{smallmatrix}\right)\right)$.

Put $W_{\pi,\epsilon}(g) = \sum_{J\subset I}\epsilon(J)W_\pi\left(g\left(\begin{smallmatrix} -1_J & 0 \\ 0 & 1\end{smallmatrix}\right)\right)$. Using (12) the integral unfolds:

$$\int_{\mathbb{A}^\times/F^\times}\phi(y)f_{\pi,1}\left(\begin{smallmatrix} y & y_p p^{-\beta} \\ 0 & 1\end{smallmatrix}\right)d^\times y = \int_{\mathbb{A}^\times}\phi(y)W_{\pi,1}\left(\begin{smallmatrix} y & y_p p^{-\beta} \\ 0 & 1\end{smallmatrix}\right)d^\times y.$$

Since $\phi_\tau = 1$ for all $\tau \in I$, we have $\int_{\mathbb{A}^\times}\phi(y)W_{\pi,\epsilon}\left(\begin{smallmatrix} y & y_p p^{-\beta} \\ 0 & 1\end{smallmatrix}\right)d^\times y = 0$ unless $\epsilon = 1$. From this and the fact that $2^d W_\pi = \sum_\epsilon W_{\pi,\epsilon}$ the above integral equals

$$2^d\int_{\mathbb{A}^\times}\phi(y)W_\pi\left(\begin{smallmatrix} y & y_p p^{-\beta} \\ 0 & 1\end{smallmatrix}\right)d^\times y = 2^d\int_{\mathbb{A}_f^\times F_\infty^+}\phi(y)W_\pi\left(\begin{smallmatrix} y & y_p p^{-\beta} \\ 0 & 1\end{smallmatrix}\right)d^\times y,$$

the last equality coming from the fact that f_π is holomorphic. The following decomposition can be found in [Bu, Thm 3.5.4] (see [D2, §7] for the normalization):

$$W_\pi\left(\begin{smallmatrix} y & y_p p^{-\beta} \\ 0 & 1\end{smallmatrix}\right) = \prod_\tau y_\tau^{\frac{w_\tau - w_0}{2}+1}e^{-2\pi y_\tau}\prod_v W_v\left(\begin{smallmatrix} y_v & y_p p^{-\beta} \\ 0 & 1\end{smallmatrix}\right),$$

and allows us to write the integral $\Omega_{\pi,\infty}\cdot$ (15) as a product $\prod_{\tau\in I}Z_\tau\prod_v Z_v$ over all places of F. The remaining part of the proof is about the computation of these local integrals. At infinite places:

$$Z_\tau = 2\int_{\mathbb{R}_+^\times}y_\tau^{\frac{w_\tau - w_0}{2}+1}e^{-2\pi y_\tau}d^\times y_\tau = \Gamma_{\mathbb{C}}\left(\frac{w_\tau - w_0}{2}+1\right). \tag{16}$$

It follows then from (11) that $\prod_\tau Z_\tau = \Gamma(\pi, 1)$.

At a finite place v the normalization of W_π is such that $W_v\left(\begin{smallmatrix} \varpi_v^n & 0 \\ 0 & 1\end{smallmatrix}\right) = 0$ for $n < -\delta_v$ and $W_v\left(\begin{smallmatrix} \varpi_v^{-\delta_v} & 0 \\ 0 & 1\end{smallmatrix}\right) = 1$.

If v does not divide p (see [Bu] for the last equality):

$$Z_v = \int_{F_v^\times}\phi_v(y_v)W_v\left(\begin{smallmatrix} y_v & 0 \\ 0 & 1\end{smallmatrix}\right)d^\times y_v = \sum_{n\in\mathbb{Z}}\phi_v(\varpi_v^n)W_v\left(\begin{smallmatrix} \varpi_v^n & 0 \\ 0 & 1\end{smallmatrix}\right) = L_v(\pi_v\otimes\phi_v, 1). \tag{17}$$

Finally for v dividing p, one has $W_v\left(\begin{smallmatrix} \varpi_v^{n-\delta_v} & 0 \\ 0 & 1\end{smallmatrix}\right) = \begin{cases}\alpha_v^n & , n\geq 0 \\ 0 & , n < 0\end{cases}$, hence

$$\begin{aligned}Z_v &= (\alpha_v\nu_v(\varpi_v))^{-e_v\beta}\int_{F_v^\times}\phi_v(y_v\varpi_v^{-e_v\beta})W_v\left(\begin{smallmatrix} y_v & y_v\varpi_v^{-e_v\beta} \\ 0 & 1\end{smallmatrix}\right)d^\times y_v = \\ &= \alpha_v^{-e_v\beta}\int_{F_v^\times}(\phi_v\nu_v)(y_v\varpi_v^{-e_v\beta})\xi_v(y_v\varpi_v^{-e_v\beta})(\nu_v^{-1}\circ\det\cdot W_v)\left(\begin{smallmatrix} y_v & 0 \\ 0 & 1\end{smallmatrix}\right)d^\times y_v = \\ &= \sum_{n\geq 0}\alpha_v^{n-e_v\beta}\int_{\mathfrak{o}_v^\times}(\phi_v\nu_v)(u\varpi_v^{n-e_v\beta-\delta_v})\xi_v(u\varpi_v^{n-e_v\beta-\delta_v})du.\end{aligned}$$

If $\phi_v\nu_v$ is ramified, then the latter integral is zero unless $n - e_v\beta = -\operatorname{cond}(\phi_v\nu_v)$, yielding $Z_v = \alpha_v^{-\operatorname{cond}(\phi_v\nu_v)}\tau_v(\phi_v\nu_v, \xi_v)$ by (13).

If $\phi_v\nu_v$ is unramified, since $\nu_v(\varpi_v) = 1$ by definition, we have:

$$\begin{aligned}
Z_v &= \phi_v(\varpi_v)^{-\delta_v} \sum_{n \geq e_v\beta - 1} (\alpha_v\phi_v(\varpi_v))^{n-e_v\beta} \int_{\mathfrak{o}_v^\times} \xi_v(u\varpi_v^{n-e_v\beta-\delta_v})du = \\
&= \phi_v(\varpi_v)^{-\delta_v}((1-\alpha_v\phi_v(\varpi_v))^{-1}(1-\mathrm{N}_{F/\mathbb{Q}}(v)^{-1}) - (\alpha_v\phi_v(\varpi_v))^{-1}\mathrm{N}_{F/\mathbb{Q}}(v)^{-1}) = \\
&= \phi_v(\varpi_v)^{-\delta_v}(1-(\alpha_v\phi_v(\varpi_v)\mathrm{N}_{F/\mathbb{Q}}(v))^{-1})(1-\alpha_v\phi_v(\varpi_v))^{-1}.
\end{aligned}$$

This together with (15), (16) and (17) completes the proof of (i).

(ii) The proof relies on the interpolation formula proved in (i). Put $\pi' = \pi \otimes |\cdot|_{\mathbb{A}}\omega^{-1}$. One has to compare the specialization of $L_p(\pi)$ by $\phi\chi\omega^{-1}$ with the specialization of $L_p(\pi')$ by ϕ. First of all, we have $\Omega_{\pi,\infty} = \Omega_{\pi',\infty}$. In fact, since $K_p \subset K_{11}(p)$, the sheaf $\mathcal{L}_K(w, w_0; \mathbb{C})$ on Y_K is canonically isomorphic to the sheaf $\mathcal{L}_K(w, w_0 - 2; \mathbb{C})$ twisted by ω^{-1}, hence b_π and $b_{\pi'}$ are basis of the same $\mathcal{O}$-line. Moreover, since the ordinary twist type of π'_v is $\nu_v\omega^{-1}$, the local factors at v dividing p are the same. Finally, using Manin's trick (see [Ki, Lemma 4.6]), one finds the same L-functions and Γ-factors. □

2.3. **Σ-stabilized p-adic L-functions.** This section is preparatory for the construction of p-adic L-functions for Hida families of Hilbert automorphic forms. The natural parameters on those families are a residual Galois representation $\bar{\rho}$ (here equal to $\rho_{\pi,p}$ mod p) and a fixed finite set Σ of primes of F not dividing p containing those dividing the tame part of the conductor of π.

It turns out that in order to be able to put the periods $\Omega_{\pi,\infty}$ in a p-adic family, one should modify them by replacing the newform f_π by a certain Σ-stabilized automorphic form f_π^Σ (see [D2, §7.1]). Such forms have already been used in [W2] and [DFG] (as well as in [Fu] for the Hilbert modular case).

2.3.1. *Tame level associated to $\bar{\rho}$ and Σ.* We define a tame level $K^p_{\bar{\rho},\Sigma} = \prod_{v \nmid p} K_v$ as follows. We denote by c_v the minimal conductor of $\bar{\rho}_v$. Let $\bar{\nu}_v$ be a minimal twist character of $\bar{\rho}_v$ and let d_v denote the dimension of inertia invariants in $\bar{\rho}_v \otimes \bar{\nu}_v^{-1}$ (see [D2, 4.1] for the terminology). For $v \in \Sigma$ let

$$K_v = \ker(K_1(v^{c_v+d_v}) \xrightarrow{\det} \mathfrak{o}_v^\times \xrightarrow{\widetilde{\nu}_v} \mathcal{O}^\times),$$

where $\widetilde{\nu}_v$ denotes the Teichmüller lift of $\bar{\nu}_v$.

To ensure the neatness of $K^p_{\bar{\rho},\Sigma}K_p$ we put $K_\mathfrak{u} = K_0(\mathfrak{u})$ for place $\mathfrak{u}$ chosen as in [D2, Lemma 2.2] and fix a root $\alpha_\mathfrak{u}$ of the Hecke polynomial at $\mathfrak{u}$. Finally, for all $v \notin \Sigma$, $v \nmid p\,\mathfrak{u}$ we put $K_v = \mathrm{GL}_2(\mathfrak{o}_v)$.

In addition to e_p, e_∞ and e_π, we introduce the following idempotent.

Definition 2.7. Denote by e_Σ is the localisation at $(U_\mathfrak{u} - \alpha_\mathfrak{u}, U_v; v \in \Sigma)$ followed by the $\widetilde{\nu}_v$-isotypic part for the action of the Hecke operators $U_\delta = \left[K_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right) K_v\right]$ for every $v \in \Sigma$ and $\delta \in \mathfrak{o}_v^\times$.

Lemma 2.8. *If $K = K^p_{\bar\rho,\Sigma}K_p$ with $K_p \supset K_{11}(p^\alpha)$ and $\mathfrak{a} = p^\beta\Sigma$ with $\beta \geq \alpha$, then the ideal $\mathfrak{a}_K$ from definition 1.13 can be chosen to divide the Artin conductor of $\bar\rho$. In particular $\mathrm{Cl}_F^{(p)}(p^\infty\Sigma\,\mathfrak{a}_K) = \mathrm{Cl}_F^{(p)}(p^\infty\Sigma)$.*

2.3.2. For every $v \in \Sigma$, there is a canonical isomorpshism:

$$\pi_v^{K_v}[\widetilde{\nu}_v] \simeq (\pi_v \otimes \widetilde{\nu}_v^{-1})^{K_1(v^{c_v+d_v})}.$$

and the latter contains a unique line on which U_v acts as 0. It follows that for α large enough

$$e_\Sigma e_\pi e_\infty e_p \,\mathrm{H}_c^d\left(Y_{K^p_{\bar\rho,\Sigma}K_{11}(p^\alpha)}, \mathcal{L}_K(w, w_0; \mathbb{C})\right)$$

is a complex line having a basis $\delta_\infty(f_\pi^\Sigma)$, where δ_∞ denotes the Matsushima-Shimura-Harder isomorphism (see [H5] or [D2, §7.1]) and f_π^Σ is an automorphic form in π which is nearly ordinary at all places dividing p, Σ-stabilized, new outside $p\Sigma$ and normalized using its adelic Fourier expansion (12). This complex line contains a natural $\mathcal{O}$-line, namely the torsion-free part of

$$e_\Sigma e_\pi e_\infty e_p \,\mathrm{H}_c^d\left(Y_{K^p_{\bar\rho,\Sigma}K_{11}(p^\alpha)}, \mathcal{L}_K(w, w_0; \mathcal{O})\right),$$

and we denote by b_π^Σ a basis of the latter.

Definition 2.9. The Archimedean period $\Omega_{\pi,\infty}^\Sigma = \delta_\infty(f_\pi^\Sigma)/b_\pi^\Sigma \in \mathbb{C}^\times$ is well defined up to an element of $\mathcal{O}^\times$.

Definition 2.10. Assume that the weight (w, w_0) is critical. The Σ-stabilized p-adic L-function $L_p^\Sigma(\pi) \in \mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)]]$ attached to π, is defined as the image of $S^{w,w_0}_{K^p_{\bar\rho,\Sigma}K_{11}(p^\alpha),\widetilde{\Sigma}}(b_\pi^\Sigma)$ (see definition 1.20), where

$$\widetilde{\Sigma} = \prod_{v\in\Sigma} v^{b_v}, \text{ with } \; b_v = \max(1, \mathrm{cond}(\bar\nu_v)). \tag{18}$$

Let ν_v and α_v for v dividing p be as in §2.2.

Theorem 2.11. (i) *Let $\phi : \mathrm{Cl}_F^+(p^\infty\Sigma) \to \mathcal{O}^\times$ be a p-power order character. Then the image of $L_p^\Sigma(\pi)$ by the resulting homomorphism $\mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)]] \to \mathcal{O}$ equals:*

$$\frac{L^{(p\Sigma)}(\pi\otimes\phi, 1)\Gamma(\pi, 1)}{\Omega_{\pi,\infty}^\Sigma}\prod_{v|p\Sigma} Z_v,$$

where for v dividing p, Z_v is as in theorem 2.5 and for v dividing Σ we have:
$$Z_v = \begin{cases} \tau_v(\phi_v\widetilde{\nu}_v, \xi_v) & \textit{if } \phi_v\widetilde{\nu}_v \textit{ is ramified,} \\ -\mathrm{N}_{F/\mathbb{Q}}(v)^{-1}\phi_v(\varpi_v)^{-1-\delta_v} & \textit{otherwise.}\end{cases}$$

(ii) *If (w, w_0-2) is critical too, the automorphism $[a] \mapsto (\chi\omega^{-1})(a)[a]$ of $\mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)]]$ sends $L_p^\Sigma(\pi)$ to $L_p^\Sigma(\pi\otimes|\cdot|_{\mathbb{A}}\omega^{-1})$.*

Proof. (i) As in the proof of theorem 2.5 the specialization of $L_p^\Sigma(\pi)$ by ϕ equals

$$\frac{\phi(\widetilde{\Sigma}^{-1})\prod_{v|p}((\phi_v\nu_v)(\varpi_v)\alpha_v)^{-e_v\beta}}{\Omega^\Sigma_{\pi,\infty}}\int_{\mathbb{A}^\times/F^\times}\phi(y)f^\Sigma_{\pi,1}\left(\begin{smallmatrix} y & y_p\Sigma p^{-\beta}\widetilde{\Sigma}^{-1}\\ 0 & 1\end{smallmatrix}\right)d^\times y.$$

The integral naturally decomposed as a product $(\Omega^\Sigma_{\pi,\infty})^{-1}\cdot\prod_\tau Z_\tau\prod_v Z_v$ over the places of F and for $v\notin\Sigma$ the computation of the local term Z_v is as in the proof of theorem 2.5. Since the automorphic form f_π^Σ is Σ-stabilized, the local component at v dividing Σ of its adelic Whittaker function W_π^Σ is given by the formula

$$W_v^\Sigma\left(\begin{smallmatrix}\varpi_v^{n-\delta_v} & 0\\ 0 & 1\end{smallmatrix}\right)=\begin{cases}1, & n=0\\ 0, & n\neq 0\end{cases}\text{, hence}$$

$$Z_v=\int_{F_v^\times}\phi_v(y_v\varpi_v^{-b_v})W_v^\Sigma\left(\begin{smallmatrix} y_v & y_v\varpi_v^{-b_v}\\ 0 & 1\end{smallmatrix}\right)d^\times y_v=\int_{\mathfrak{o}_v^\times}(\phi_v\widetilde{\nu}_v)(u\varpi_v^{-\delta_v-b_v})\xi_v(u\varpi_v^{-\delta_v-b_v})du.$$

If $\phi_v\widetilde{\nu}_v$ is ramified, then $b_v=\mathrm{cond}(\phi_v\widetilde{\nu}_v)$ and $Z_v=\tau_v(\phi_v\widetilde{\nu}_v,\xi_v)$ by (13). Othewise ϕ_v and $\widetilde{\nu}_v$ have to be both unramified, hence $b_v=1$ and $Z_v=-\mathrm{N}_{F/\mathbb{Q}}(v)^{-1}\phi_v(\varpi_v)^{-1-\delta_v}$.

(ii) is proved as in theorem 2.5. □

We will see that the p-adic L-functions $L_p^\Sigma(\pi)$ behave well when π varies in a Hida family.

3. Exact control theorem for the nearly ordinary cohomology of Hilbert modular varieties

The main result in this section is an exact control for the nearly ordinary Betti cohomology of a Hilbert modular variety with coefficients in $\mathcal{O}$, after localization at a certain maximal ideal of the Hecke algebra. The proofs rely on results established in [D1, D2] on the (absence of) torsion in these cohomology groups under the assumptions ($\star$) and ($\star\star$), hence rely indirectly on the fact that the Hilbert modular varieties admit a canonical model over $\overline{\mathbb{Q}}$ allowing to interpret Betti cohomology with p-adic coefficients as etale cohomology and further as de Rham cohomology after extending the scalars to a Fontaine ring of periods.

Henceforth p is assumed to be odd.

3.1. **Towers of Hilbert modular varieties.** Since we are mostly interested in p-adic cohomological interpolation, we will fix a tame level K^p and vary the level at p.

For $?\in\{0,1,11\}$ we denote by $Y_?(p^\alpha)$ the Hilbert modular variety of level $K^pK_?(p^\alpha)$ and by $Y_?^{\mathrm{ad}}(p^\alpha)$ its adjoint counterpart. By [D2, Lemmas 2.1,2.2] the groups of the abelian coverings

$$Y_{11}(p^\alpha)\to Y_0^{\mathrm{ad}}(p^\alpha),\ Y_1(p^\alpha)\to Y_0^{\mathrm{ad}}(p^\alpha)\text{ and }Y_{11}^{\mathrm{ad}}(p^\alpha)\to Y_0^{\mathrm{ad}}(p^\alpha)$$

are respectively isomorphic to

$$\mathbb{A}_f^\times K_0(p^\alpha)/F^\times K_{11}(p^\alpha),\ \mathbb{A}_f^\times K_0(p^\alpha)/F^\times K_1(p^\alpha)\text{ and }K_0(p^\alpha)/K_{11}(p^\alpha)(\mathfrak{o}\otimes\mathbb{Z}_p)^\times.$$

We identify the standard torus of GL_2 with $\mathbb{G}_m^2$ via $(u,z) \mapsto \left(\begin{smallmatrix} uz & 0 \\ 0 & z \end{smallmatrix}\right)$. This leads to the following identifications:

$\mathbb{A}_f^\times K_0(p^\alpha)/F^\times K_1(p^\alpha) \simeq \mathbb{A}_f^\times/F^\times(\mathbb{A}_f^\times \cap K^p K_1(p^\alpha))$ and $K_0(p^\alpha)/K_{11}(p^\alpha)(\mathfrak{o}\otimes\mathbb{Z}_p)^\times \simeq (\mathfrak{o}/p^\alpha)^\times$,

$$\text{hence } \mathbb{A}_f^\times K_0(p^\alpha)/F^\times K_{11}(p^\alpha) \simeq \mathbb{A}_f^\times/F^\times(\mathbb{A}_f^\times \cap K^p K_1(p^\alpha)) \times (\mathfrak{o}/p^\alpha)^\times.$$

We define the following semi-local p-adic Iwasawa $\mathcal{O}$-algebras:

$$\Lambda_1 = \mathcal{O}\left[\left[\varprojlim_{\alpha\geq 1} \mathbb{A}_f^\times/F^\times(\mathbb{A}_f^\times \cap K^p K_1(p^\alpha))\right]\right] = \mathcal{O}\left[\left[z; z \in \mathbb{A}_f^\times/\overline{F^\times(\mathbb{A}_f^\times \cap K^p)}\right]\right],$$

$$\Lambda_{\mathrm{ad}} = \mathcal{O}\left[\left[\varprojlim_{\alpha\geq 1} (\mathfrak{o}/p^\alpha)^\times\right]\right] = \mathcal{O}[[u; u \in (\mathfrak{o}\otimes\mathbb{Z}_p)^\times]] \text{ and } \Lambda_{11} = \Lambda_1 \widehat{\otimes} \Lambda_{\mathrm{ad}}.$$

3.2. **Hida's stabilization lemma.** For a cohomological weight (w, w_0) define:

$$\mathcal{H}_{11}(w,w_0) = \mathrm{Hom}_{\mathcal{O}}\left(\varinjlim_{\alpha\geq 1} e_p \, \mathrm{H}_c^\bullet(Y_{11}(p^\alpha), \mathcal{L}(w,w_0; E/\mathcal{O})), E/\mathcal{O}\right),$$

$$\mathcal{H}_{1}(w,w_0) = \mathrm{Hom}_{\mathcal{O}}\left(\varinjlim_{\alpha\geq 1} e_p \, \mathrm{H}_c^\bullet(Y_{1}(p^\alpha), \mathcal{L}(w,w_0; E/\mathcal{O})), E/\mathcal{O}\right) \text{ and}$$

$$\mathcal{H}_{\mathrm{ad}}(w,w_0) = \mathrm{Hom}_{\mathcal{O}}\left(\varinjlim_{\alpha\geq 1} e_p \, \mathrm{H}_c^\bullet(Y_{11}^{\mathrm{ad}}(p^\alpha), \mathcal{L}(w,w_0; E/\mathcal{O})), E/\mathcal{O}\right),$$

where e_p denotes Hida's (nearly) ordinary idempotent. It follows from §3.1 that $\mathcal{H}_?(w,w_0)$ is naturally a $\Lambda_?$-module, for $? \in \{11, 1, \mathrm{ad}\}$.

By Hida's stabilization lemma $\mathcal{H}_?(w,w_0)$ is independent of (w,w_0) in the following sense:

Theorem 3.1. *Let $(w,w_0) \in \mathbb{N}[I]\times\mathbb{Z}$ be a cohomological weight.*

(i) *(*[H3, (3.3)]*) There is an isomorphism of Λ_{11}-modules*

$$\mathcal{H}_{11}(w,w_0) \simeq \mathcal{H}_{11}(0,0) \otimes_{\mathcal{O}} \mathcal{O}\, Y^w,$$

where $[u,z] \in \Lambda_{11}$ acts by $u^{\frac{w_0 t - w}{2}} \chi(z)^{w_0}\omega(z)^{-w_0}$ on the lowest weight vector $Y^w \in \mathcal{L}(w,w_0;\mathcal{O})$.

(ii) *(*[H1, Thm 8.6]*) There is an isomorphism of Λ_1-modules*

$$\mathcal{H}_1(w+t, w_0+1) \simeq \mathcal{H}_1(w,w_0) \otimes_{\mathcal{O}} \mathcal{O}\, Y,$$

where $[z] \in \Lambda_1$ acts Y by $\chi(z)\omega(z)^{-1}$.

(iii) *There is an isomorphism of Λ_{ad}-modules*

$$\mathcal{H}_{\mathrm{ad}}(w,w_0) \simeq \mathcal{H}_{\mathrm{ad}}(|w_0|t, w_0) \otimes_{\mathcal{O}} \mathcal{O}\, Y^{w-|w_0|t},$$

where $[u] \in \Lambda_{\mathrm{ad}}$ acts on $Y^{w-|w_0|t}$ by $u^{\frac{|w_0|t-w}{2}}$.

3.3. **Exact control theorem.** Our exact control theorem will only hold after applying a certain idempotent $e_{\bar\rho}$, analogous the Mazur's non-Eisenstein idempotent, that we will now define. Given a finite set of primes Σ outside p and a continuous representation $\bar\rho : \mathrm{Gal}_{F,p\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ we consider the following maximal ideal:

$$\mathfrak{m}_{\bar\rho} = \big(\varpi, T_v - \mathrm{Tr}(\bar\rho(\mathrm{Frob}_v)), S_v - \mathrm{Det}(\bar\rho(\mathrm{Frob}_v))\,\mathrm{N}_{F/\mathbb{Q}}(v)\big)$$

of the abstract Hecke algebra $\mathbb{T}^\Sigma = \mathcal{O}[T_v, S_v\ ;\ v \notin \Sigma, v \nmid p]$, where ϖ denotes a uniformizer of $\mathcal{O}$.

In addition to its $\Lambda_?$-module structure, $\mathcal{H}_?(w, w_0)$ is endowed with an action of $\mathbb{T}^\Sigma$ and of U_v and $U_\delta = [K_v \left(\begin{smallmatrix}1&0\\0&\delta\end{smallmatrix}\right) K_v]$ for every v dividing p and $\delta \in \mathfrak{o}_v^\times$.

Definition 3.2. Assume that $\bar\rho$ satisfies $(\star\star)$, so that there exists a cohomological cuspidal automorphic representation $\bar\pi$ of weight $(\bar w, \bar w_0)$ which is nearly ordinary at v dividing p with U_v^0-eigenvalue α_v^0, and such that $\bar\rho \simeq \rho_{\bar\pi,p} \mod p$.

Define $e_{\bar\rho}$ as the idempotent corresponding to the localization at

$$(\mathfrak{m}_{\bar\rho}, U_v^0 - \alpha_v^0, U_\delta - \delta^{(\bar w - \bar w_0 t - w + w_0 t)/2}; v \mid p, \delta \in \mathfrak{o}_v^\times).$$

Remark 3.3. The weight of $\bar\pi$ as in $(\star\star)$ is denoted $(\bar w, \bar w_0)$ since it only depends on $\bar\rho$. In fact by remark 4.2 for all primes v dividing p, the nearly ordinariness of π_v implies the reducibility of $\rho_{\bar\pi,p}|_{\mathrm{Gal}_{F_v}}$ (hence of $\bar\rho|_{\mathrm{Gal}_{F_v}}$) and the fact that the weights are smaller than $p-1$ allows them to be recovered from $\bar\rho$. On the other hand, the reduction α_v^0 (mod ϖ) cannot be retrieved from $\bar\rho$ in general, hence there is a slight abuse in the notation $e_{\bar\rho}$.

Definition 3.4. For a cohomological weight (w, w_0) define:

$$\mathcal{H}^{\mathrm{n.o}}_{\bar\rho}(w, w_0) = \mathrm{Hom}_{\mathcal{O}}\left(\varinjlim_{\alpha\geq 1} e_{\bar\rho}\,\mathrm{H}^\bullet_c(Y_{11}(p^\alpha), \mathcal{L}(w, w_0; E/\mathcal{O})), E/\mathcal{O}\right),$$

$$\mathcal{H}^{\mathrm{ord}}_{\bar\rho}(w, w_0) = \mathrm{Hom}_{\mathcal{O}}\left(\varinjlim_{\alpha\geq 1} e_{\bar\rho}\,\mathrm{H}^\bullet_c(Y_{1}(p^\alpha), \mathcal{L}(w, w_0; E/\mathcal{O})), E/\mathcal{O}\right) \text{ and}$$

$$\mathcal{H}^{\mathrm{det}}_{\bar\rho}(w, w_0) = \mathrm{Hom}_{\mathcal{O}}\left(\varinjlim_{\alpha\geq 1} e_{\bar\rho}\,\mathrm{H}^\bullet_c(Y^{\mathrm{ad}}_{11}(p^\alpha), \mathcal{L}(w, w_0; E/\mathcal{O})), E/\mathcal{O}\right).$$

Remark 3.5. The localization at $(U_\delta - \delta^{(\bar w - \bar w_0 t - w + w_0 t)/2}; v \mid p, \delta \in \mathfrak{o}_v^\times)$ is superfluous for the definition of $\mathcal{H}^{\mathrm{ord}}_{\bar\rho}(w, w_0)$, but not *a priori* for $\mathcal{H}^{\mathrm{det}}_{\bar\rho}(w, w_0)$ and $\mathcal{H}^{\mathrm{n.o}}_{\bar\rho}(w, w_0)$.

We define the local p-adic Iwasawa $\mathcal{O}$-algebra $\Lambda^{\mathrm{n.o}} = \Lambda^{\mathrm{ord}} \widehat{\otimes} \Lambda^{\mathrm{det}}$, where

$$\Lambda^{\mathrm{ord}} = \mathcal{O}\left[\left[z; z \in \left(\mathbb{A}_f^\times / \overline{F^\times(\mathbb{A}_f^\times \cap K^p)}\right)^{p-\mathrm{part}}\right]\right] \text{ and } \Lambda^{\mathrm{det}} = \mathcal{O}\left[\left[u; u \in (\mathfrak{o}\otimes\mathbb{Z}_p)^{\times p-\mathrm{part}}\right]\right].$$

By global class field theory, the character $\det(\bar\rho)\cdot\bar\chi : \mathrm{Gal}_{F,p\Sigma} \to \overline{\mathbb{F}}_p^\times$ factors through the prime to p-part of the class group $\mathbb{A}_f^\times / \overline{F^\times(\mathbb{A}_f^\times \cap K^p)}$, hence its Teichmüller lift $\widetilde{\det(\bar\rho)}\omega$ induces a surjective homomorphism $\Lambda_1 \twoheadrightarrow \Lambda^{\mathrm{ord}}$.

The Teichmüller lift of the character $(\mathfrak{o}\otimes\mathbb{Z}_p)^\times \to \overline{\mathbb{F}}_p^\times, u\mapsto \bar{u}^{(\bar{w}-\bar{w}_0t-w+w_0t)/2}$ induces a surjective homomorphism $\Lambda_{\mathrm{ad}}\twoheadrightarrow\Lambda^{\mathrm{det}}$.

From the above two surjective homomorphisms one deduces a third one:

$$\Lambda^{\mathrm{n.o}}=\Lambda^{\mathrm{ord}}\widehat{\otimes}\Lambda^{\mathrm{det}}\twoheadrightarrow\Lambda_{11}=\Lambda_1\widehat{\otimes}\Lambda_{\mathrm{ad}}.$$

The idempotent $e_{\bar\rho}$ determines residually the eigenvalues of S_v, for v outside a finite set, hence by weak approximation determines the central action residually. Moreover, it imposes residually the action of U_δ for all $\delta\in(\mathfrak{o}\otimes\mathbb{Z}_p)^\times$. Since the residual characteristic is p, it follows that the idempotent $e_{\bar\rho}$ fixes the action of the prime to p parts of $(\mathfrak{o}\otimes\mathbb{Z}_p)^\times$ and of the class group $\mathbb{A}_f^\times/\overline{F^\times(\mathbb{A}_f^\times\cap K^p)}$. We record this fact in the following proposition.

Lemma 3.6. (i) *The action of* Λ_1 *on* $\mathcal{H}^{\mathrm{ord}}_{\bar\rho}(w,w_0)$ *is via the above* $\Lambda_1\twoheadrightarrow\Lambda^{\mathrm{ord}}$.

(ii) *The action of* Λ_{ad} *on* $\mathcal{H}^{\mathrm{det}}_{\bar\rho}(w,w_0)$ *is via the above* $\Lambda_{\mathrm{ad}}\twoheadrightarrow\Lambda^{\mathrm{det}}$.

(iii) *The action of* Λ_{11} *on* $\mathcal{H}^{\mathrm{n.o}}_{\bar\rho}(w,w_0)$ *is via the above* $\Lambda_{11}\twoheadrightarrow\Lambda^{\mathrm{n.o}}$.

Definition 3.7. For $\alpha\geq 1$ and $?\in\{\mathrm{n.o},\mathrm{ord},\mathrm{det}\}$ denote by $P^?_\alpha\subset\Lambda^?$ the kernel of the homomorphism induced by $[u,z]\mapsto[u \mod p^\alpha, z \mod p^\alpha]$.

Theorem 3.8. *Suppose that* ($\star$) *and* ($\star\star$) *hold.*

(i) *For* $?\in\{\mathrm{n.o},\mathrm{ord},\mathrm{det}\}$ *and for all* $\alpha\geq1$ *we have exact control:*

$$\mathcal{H}^?_{\bar\rho}(\bar w,\bar w_0)\underset{\Lambda^?}{\otimes}\Lambda^?/P^?_\alpha\simeq\mathrm{Hom}_{\mathcal{O}}(e_{\bar\rho}\,\mathrm{H}^d(Y_?(p^\alpha),\mathcal{L}(\bar w,\bar w_0;E/\mathcal{O})),E/\mathcal{O}).$$

(ii) *For* $?\in\{\mathrm{n.o},\mathrm{ord},\mathrm{det}\}$ *the* $\Lambda^?$*-module* $\mathcal{H}^?_{\bar\rho}(\bar w,\bar w_0)$ *is free of finite rank.*

(iii) *Given a cohomological weight* (w,w_0)*, for all* $\alpha\geq1$ *we have exact control:*

$$\mathcal{H}^{\mathrm{n.o}}_{\bar\rho}(w,w_0)\underset{\Lambda^{\mathrm{n.o}}}{\otimes}\Lambda^{\mathrm{n.o}}/P^{\mathrm{n.o}}_\alpha\simeq\mathrm{Hom}_{\mathcal{O}}(e_{\bar\rho}\,\mathrm{H}^d(Y_{11}(p^\alpha),\mathcal{L}(w,w_0;E/\mathcal{O})),E/\mathcal{O}).$$

(iv) *Given a cohomological weight* $(w,w_0)\in(\bar w,\bar w_0)+\mathbb{Z}(t,1)$*, for all* $\alpha\geq1$ *we have exact control:*

$$\mathcal{H}^{\mathrm{ord}}_{\bar\rho}(w,w_0)\underset{\Lambda^{\mathrm{ord}}}{\otimes}\Lambda^{\mathrm{ord}}/P^{\mathrm{ord}}_\alpha\simeq\mathrm{Hom}_{\mathcal{O}}(e_{\bar\rho}\,\mathrm{H}^d(Y_1(p^\alpha),\mathcal{L}(w,w_0;E/\mathcal{O})),E/\mathcal{O}).$$

(v) *If* $(w,\bar w_0)$ *is a cohomological weight then for all* $\alpha\geq1$ *we have exact control:*

$$\mathcal{H}^{\mathrm{det}}_{\bar\rho}(w,\bar w_0)\underset{\Lambda^{\mathrm{det}}}{\otimes}\Lambda^{\mathrm{det}}/P^{\mathrm{det}}_\alpha\simeq\mathrm{Hom}_{\mathcal{O}}(e_{\bar\rho}\,\mathrm{H}^d(Y^{\mathrm{ad}}_{11}(p^\alpha),\mathcal{L}(w,\bar w_0;E/\mathcal{O})),E/\mathcal{O}).$$

Proof. (i) follows from Hida's exact control criterion [H6, lemma 7.1] and the fact that the Pontryagin dual of $e_{\bar\rho}\,\mathrm{H}^d(Y_{K^p\,\mathrm{GL}_2(\mathfrak{o}\otimes\mathbb{Z}_p)},\mathcal{L}(\bar w,\bar w_0;E/\mathcal{O})$ is isomorphic to the torsion free $\mathcal{O}$-module $e_{\bar\rho}\,\mathrm{H}^d(Y_{K^p\,\mathrm{GL}_2(\mathfrak{o}\otimes\mathbb{Z}_p)},\mathcal{L}(\bar w,\bar w_0;\mathcal{O}))$ (see [D2, §2]).

(ii) follows from (i) by a commutative algebra argument as in the proof of [MT, Thm 9]

(iii),(iv) and (v) follows from (i) together with theorem 3.1, as in the last paragraph of the proof of [H6, Thm 7.1]. □

As a corollary we obtain the freeness of the nearly ordinary part of the cohomology of a Hilbert modular variety without assuming that it has good reduction at p and that the weight of the local system is p-small, thus generalizing [D2, Theorem 2.3].

Corollary 3.9. *If $\bar{\rho}$ satisfies $(\star)$ and $(\star\star)$, then $e_{\bar{\rho}}\,\mathrm{H}^d(Y_{11}(p^\alpha), \mathcal{L}(w, w_0; \mathcal{O}))$ is a free $\mathcal{O}$-module of finite rank, whose Pontryagin dual is $e_{\bar{\rho}}\,\mathrm{H}^d(Y_{11}(p^\alpha), \mathcal{L}(w, w_0; E/\mathcal{O}))$.*

4. Freeness and $R = T$ theorems

4.1. **Universal nearly ordinary deformation rings.** Let Σ be a finite set of primes outside p and let $\bar{\rho} : \mathrm{Gal}_{F,p\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous irreducible representation which is nearly ordinary and distinguished at p, meaning for every v dividing p there exist two distinct characters $\bar{\psi}_{1,v}$ and $\bar{\psi}_{2,v}$ of the absolute Galois group Gal_{F_v} of F_v, such that

$$\bar{\rho}_{|\mathrm{Gal}_{F_v}} \simeq \begin{pmatrix} \bar{\psi}_{1,v} & * \\ 0 & \bar{\psi}_{2,v} \end{pmatrix}.$$

Let $\mathcal{R}^{\mathrm{n.o}}_{\bar{\rho},\Sigma}$ be Mazur's universal $\mathcal{O}$-algebra parametrizing deformations ρ' of $\bar{\rho}$ which are nearly ordinary at all v dividing p, in the sense that $\rho'_{|\mathrm{Gal}_{F_v}} \simeq \begin{pmatrix} \psi'_{1,v} & * \\ 0 & \psi'_{2,v} \end{pmatrix}$, where $\psi'_{i,v}$ is a lift of $\bar{\psi}_{i,v}$ $(i = 1, 2)$.

Denote by $\rho_{\mathcal{R}}$ the universal deformation. For every v dividing p, by nearly ordinariness, $\rho_{\mathcal{R}|\mathrm{Gal}_{F_v}} \simeq \begin{pmatrix} \psi'_{1,v} & * \\ 0 & \psi'_{2,v} \end{pmatrix}$, hence by local class field theory $\psi'_{2,v}\widetilde{\psi}_{2,v}^{-1}$ is a character of $\mathfrak{o}_v^{\times p-\mathrm{part}}$, where $\widetilde{\psi}_{2,v}$ denotes the Teichmüller lift of $\bar{\psi}_{2,v}$. This endows $\mathcal{R}^{\mathrm{n.o}}_{\bar{\rho},\Sigma}$ with $\mathcal{O}[[(\mathfrak{o} \otimes \mathbb{Z}_p)^{\times p-\mathrm{part}}]]$-algebra structure. It comes from the forgetful functor taking the restriction to decomposition groups at primes dividing p.

The character $\det(\rho_{\mathcal{R}})\widetilde{\det(\bar{\rho})}^{-1}\chi\omega^{-1} : \mathrm{Gal}_{F,p\Sigma} \to \mathcal{R}^{\mathrm{n.o}\,\times}_{\bar{\rho},\Sigma}$ factors through $\mathrm{Gal}(F^{(p\Sigma)}/F)$, hence endows $\mathcal{R}^{\mathrm{n.o}}_{\bar{\rho},\Sigma}$ with $\mathcal{O}[[\mathrm{Gal}(F^{(p\Sigma)}/F)]]$-algebra structure. It comes from the forgetful functor taking the determinant twisted by χ.

Therefore $\mathcal{R}^{\mathrm{n.o}}_{\bar{\rho},\Sigma}$ is naturally a module over the complete local Iwasawa algebra

$$\Lambda = \mathcal{O}[[\mathrm{Gal}(F^{(p\Sigma)}/F) \times (\mathfrak{o} \otimes \mathbb{Z}_p)^{\times p-\mathrm{part}}]],$$

which appears naturally in the class field theory of F.

Definition 4.1. An $\mathcal{O}$-algebra homomorphism $\Lambda \to \overline{\mathbb{Q}}_p$ is *algebraic* if for some cohomological weight (w, w_0) its restriction to $\mathrm{Gal}(F^{(p\Sigma)}/F)$ (resp. to $(\mathfrak{o} \otimes \mathbb{Z}_p)^{\times p-\mathrm{part}}$) is the product of a finite order character with the character χ^{-w_0} (resp. $x \mapsto x^{(w-w_0t)/2}$).

4.2. **Universal nearly ordinary Hecke rings.** Following Fujiwara [Fu] we define the universal nearly ordinary Hecke algebra as the maximal $\Lambda \otimes \mathbb{T}^\Sigma$-algebra $\mathbb{T}^{\mathrm{n.o}}_{\bar{\rho},\Sigma}$ with the following property: any homomorphism $\mathbb{T}^{\mathrm{n.o}}_{\bar{\rho},\Sigma} \to \overline{\mathbb{Q}}_p$ whose restriction to Λ is algebraic of weight (w, w_0) comes from a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$ which is cohomological of weight (w, w_0), nearly ordinary at all places dividing p and such that $\rho_{\pi,p}$ is a deformation of $\bar{\rho}$.

By a theorem of Hida [H2] and Wiles [W1], for all primes v dividing p, the nearly ordinariness of π_v implies the reducibility of $\rho_{\pi,p}|_{\mathrm{Gal}_{F_v}}$. The converse is a theorem of Saito [Sa].

Remark 4.2. If π_v is ordinary and unramified, then there is a more precise statement, namely, $\rho_{\pi,p}|_{\mathrm{Gal}_{F_v}} \simeq \begin{pmatrix} \psi_{1,v} & * \\ 0 & \psi_{2,v} \end{pmatrix}$ where (via local class field theory) the restrictions of $(\psi_{1,v})_{v|p}$ and $(\psi_{2,v})_{v|p}$ to inertia groups at primes dividing p are given by:

$$(\psi_{1,v})_{v|p} : (\mathfrak{o} \otimes \mathbb{Z}_p)^\times \to \overline{\mathbb{Z}}_p^\times, \ \ x \mapsto x^{\frac{w_0 t + w}{2} + 1} \text{ and}$$
$$(\psi_{2,v})_{v|p} : (\mathfrak{o} \otimes \mathbb{Z}_p)^\times \to \overline{\mathbb{Z}}_p^\times, \ \ x \mapsto x^{\frac{w_0 t - w}{2}}.$$

Wiles' method of pseudo representations yields a deformation of $\bar\rho$:

$$\rho_{\mathcal{R}} : \mathrm{Gal}_{F,p\Sigma} \to \mathrm{GL}_2(\mathbb{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}),$$

hence, by universal property, a surjective Λ-algebra homomorphism $\mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma} \twoheadrightarrow \mathbb{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}$.

Theorem 4.3. (Fujiwara [Fu]) *Assume that $\bar\rho$ is distinguished and that its restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}/F(\zeta_p))$ is irreducible. Then the natural surjection $\mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma} \twoheadrightarrow \mathbb{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}$ is an isomorphism of algebras which are finite flat of complete intersection over Λ.*

4.3. **Nearly ordinary cohomology modules.** We will give a concrete realization of the universal nearly ordinary Hecke algebras encountered in §4.2 by the method of p-adic cohomological interpolation described in §3.

We will first realize the groups $\mathrm{Gal}(F^{(p\Sigma)}/F) \simeq \mathrm{Cl}_F^{(p)}(p^\infty\Sigma)$ and $(\mathfrak{o} \otimes \mathbb{Z}_p)^{\times p-\mathrm{part}}$ as groups of towers of Hilbert modular varieties (see §3.1). For this purpose we fix $\bar\rho$ as in §4.1 and choose the tame level $K^p = K^p_{\bar\rho,\Sigma}$. In addition to the Hilbert modular varieties $Y_?(p^\alpha)$ and $Y_?^{\mathrm{ad}}(p^\alpha)$ defined in §2.3.1, we denote by $Y_{\bar\rho,\Sigma}$ the Hilbert modular variety of level $\mathrm{GL}_2(\mathfrak{o} \otimes \mathbb{Z}_p)K^p_{\bar\rho,\Sigma}$ which has good reduction at p.

Since the intersection of $\mathbb{A}_f^\times \cap K_1(p^\alpha)K^p_{\bar\rho,\Sigma}$ and $U(p^\alpha\Sigma)$ has prime to p index in each of them, it follows that

$$\left(\mathbb{A}_f^\times K_0(p^\alpha)/F^\times K_1(p^\alpha)\right)^{p-\mathrm{part}} = \mathrm{Cl}_F^{(p)}(p^\alpha\Sigma), \text{ hence}$$

$$\Lambda^{\mathrm{ord}} = \mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)]],\ \Lambda^{\mathrm{det}} = \mathcal{O}[[(\mathfrak{o} \otimes \mathbb{Z}_p)^{\times p-\mathrm{part}}]] \text{ and } \Lambda^{\mathrm{n.o}} = \Lambda^{\mathrm{ord}} \widehat{\otimes} \Lambda^{\mathrm{det}} \simeq \Lambda.$$

There are exact sequences:

$$1 \to (\mathfrak{o} \otimes \mathbb{Z}_p)^\times/\overline{E(\Sigma)} \to \mathrm{Cl}_F^+(p^\infty\Sigma) \to \mathrm{Cl}_F^+(\Sigma) \to 1,$$
$$1 \to \prod_{v\in\Sigma} (\mathfrak{o}/v)^\times \to \mathrm{Cl}_F^+(p^\infty\Sigma) \to \mathrm{Cl}_F^+(p^\infty) \to 1,$$

that remain exact after taking pro-p parts.

Definition 4.4. Consider the idempotent $e = e_{\bar\rho} \cdot e_\infty \cdot e_\Sigma$, with e_∞ as in definition 1.18, e_Σ as in definition 2.7 and $e_{\bar\rho}$ as in definition 3.2. For a cohomological weight (w, w_0) define:

$$\mathcal{H}^{\mathrm{n.o}}_{\bar\rho,\Sigma}(w,w_0) = \mathrm{Hom}_{\mathcal{O}}\left(\varinjlim_{\alpha\geq 1} e\,\mathrm{H}_c^\bullet(Y_{11}(p^\alpha), \mathcal{L}(w,w_0;E/\mathcal{O})), E/\mathcal{O}\right),$$

$$\mathcal{H}^{\mathrm{ord}}_{\bar\rho,\Sigma}(w,w_0) = \mathrm{Hom}_{\mathcal{O}}\left(\varinjlim_{\alpha\geq 1} e\,\mathrm{H}_c^\bullet(Y_1(p^\alpha), \mathcal{L}(w,w_0;E/\mathcal{O})), E/\mathcal{O}\right) \text{ and}$$

$$\mathcal{H}^{\det}_{\bar\rho,\Sigma}(w,w_0)=\mathrm{Hom}_{\mathcal{O}}\left(\varinjlim_{\alpha\geq 1} e\,\mathrm{H}^\bullet_c(Y^{\mathrm{ad}}_{11}(p^\alpha),\mathcal{L}(w,w_0;E/\mathcal{O})),E/\mathcal{O}\right).$$

By lemma 3.6, for any $?\in\{\mathrm{n.o},\mathrm{ord},\det\}$, $\mathcal{H}^?_{\bar\rho,\Sigma}(w,w_0)$ is endowed with a structure of $\Lambda^?$-algebra.

Definition 4.5. For a cohomological weight (w,w_0) and $?\in\{\mathrm{ord},\det,\mathrm{n.o}\}$ let $\mathcal{T}^?_{\bar\rho,\Sigma}(w,w_0)$ denote the $\Lambda^?$-algebra generated by $\mathbb{T}^\Sigma$ acting on $\mathcal{H}^?_{\bar\rho,\Sigma}(w,w_0)$.

4.4. **Freeness theorem.** Put $\mathcal{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}=\mathcal{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}(0,0)$ and $\mathcal{H}^{\mathrm{n.o}}_{\bar\rho,\Sigma}=\mathcal{H}^{\mathrm{n.o}}_{\bar\rho,\Sigma}(0,0)$.

Theorem 4.6. *Assume ($\star$) and ($\star\star$). Then $\mathcal{H}^{\mathrm{n.o}}_{\bar\rho,\Sigma}$ is free of rank one over $\mathcal{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}$ and there exist a natural isomorphism $\mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma}\xrightarrow{\sim}\mathcal{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}$ of algebras which are finite flat of complete intersection over $\Lambda^{\mathrm{n.o}}$.*

Proof. The proof follows the same strategy as in [Ti]. By theorems 3.1(i) and 3.8(iii) any homomorphism $\mathcal{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}\to\overline{\mathbb{Q}}_p$ whose restriction to $\Lambda^{\mathrm{n.o}}$ is algebraic of weight (w,w_0) comes from a cuspidal automorphic representation contributing to $e\,\mathrm{H}^d(Y_{11}(p^\alpha),\mathcal{L}(w,w_0;\mathcal{O}))$. It follows from the universal property of $\mathbb{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}$ defined in §4.2 that there exists $\Lambda^{\mathrm{n.o}}$-linear surjective homomorphism $\mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma}\twoheadrightarrow\mathbb{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}\twoheadrightarrow\mathcal{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}$.

The proof then proceeds by specialization to weight $(\bar w,\bar w_0)$. Denote by $\mathcal{P}$ the kernel of the algebraic homomorphism $\Lambda^{\mathrm{n.o}}\to\mathcal{O}$ induces by the following character:

$$\mathrm{Cl}^{(p)}_F(p^\infty\Sigma)\times(\mathfrak{o}\otimes\mathbb{Z}_p)^{\times p-\mathrm{part}}\to\mathcal{O}^\times,\ (z,u)\mapsto u^{(\bar w_0 t-\bar w)/2}\left(\det(\rho_{\bar\pi,p})\widetilde{\det(\bar\rho)}^{-1}\chi\omega^{-1}\right)(z).$$

Then $R_{\bar\rho,\Sigma}=\mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma}/\mathcal{P}\,\mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma}$ classifies deformations ρ of $\bar\rho$ of determinant $\det(\rho_{\bar\pi,p})$ whose restriction to decomposition groups at primes dividing p is as in remark 4.2. Since for all $\tau\in I$, $\bar w_\tau>0$, it is then a standard fact from Fontaine's theory that ρ is cristalline at all primes dividing p.

By the control theorem 3.8 there is a Hecke equivariant isomorhism of free $\mathcal{O}$-modules:

$$\mathcal{H}^{\mathrm{n.o}}_{\bar\rho,\Sigma}/\mathcal{P}\,\mathcal{H}^{\mathrm{n.o}}_{\bar\rho,\Sigma}\simeq e\,\mathrm{H}^d(Y_0(p),\mathcal{L}(\bar w,\bar w_0;\mathcal{O})),$$

where $Y_0(p)$ denotes the Hilbert modular variety of level $K_0(p)K^p_{\bar\rho,\Sigma}$. It is well known that if π is a cuspidal automorphic representation of cohomological weight strictly bigger than 0 such that π_v is nearly ordinary and $\pi_v^{K_0(v)}\neq 0$, then π_v is unramified. Hence:

$$e\,\mathrm{H}^d(Y_0(p),\mathcal{L}(\bar w,\bar w_0;\mathcal{O}))\simeq e\,\mathrm{H}^d(Y_{\bar\rho,\Sigma},\mathcal{L}(\bar w,\bar w_0;\mathcal{O})).$$

Then one has a commutative diagram:

$$\begin{tikzcd}
\mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma}/\mathcal{P}\,\mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma} \arrow[r, two heads] \arrow[d, equal] & \mathcal{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}/\mathcal{P}\,\mathcal{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma} \arrow[r] \arrow[d, two heads] & \mathrm{End}_{\Lambda^{\mathrm{n.o}}/\mathcal{P}\Lambda^{\mathrm{n.o}}}\left(\mathcal{H}^{\mathrm{n.o}}_{\bar\rho,\Sigma}/\mathcal{P}\,\mathcal{H}^{\mathrm{n.o}}_{\bar\rho,\Sigma}\right) \arrow[d, equal] \\
R_{\bar\rho,\Sigma} \arrow[r, two heads] & T_{\bar\rho,\Sigma} \arrow[r, hook] & \mathrm{End}_{\mathcal{O}}\left(e\,\mathrm{H}^d(Y_{\bar\rho,\Sigma},\mathcal{L}(\bar w,\bar w_0;\mathcal{O}))\right)
\end{tikzcd},$$

where $T_{\bar\rho,\Sigma}$ denotes the image of $\mathbb{T}^\Sigma$ acting on $e\,\mathrm{H}^d(Y_{\bar\rho,\Sigma},\mathcal{L}(\bar w,\bar w_0;\mathcal{O}))$.

Under the assumptions ($\star$) and ($\star\star$), we are now in position to apply [D2, Thm 6.6] and deduce that $R_{\bar\rho,\Sigma} \simeq T_{\bar\rho,\Sigma}$ and that $e\,\mathrm{H}^d(Y_{\bar\rho,\Sigma}, \mathcal{L}(\bar w, \bar w_0; \mathcal{O}))$ is free of rank one over $R_{\bar\rho,\Sigma}$. By the above diagram, it follows that $\mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma} / \mathcal{P}\,\mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma}$ is isomorphic to $\mathcal{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma} / \mathcal{P}\,\mathcal{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}$ and $\mathcal{H}^{\mathrm{n.o}}_{\bar\rho,\Sigma} / \mathcal{P}\,\mathcal{H}^{\mathrm{n.o}}_{\bar\rho,\Sigma}$ is free of rank one. It is enough then to apply Nakayama's lemma to deduce the desired result (see [Ti, 3.2] for details). $\square$

Corollary 4.7. *Exact control holds for $\mathcal{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}(w, w_0)$, that is for all $\alpha \geq 1$:*

$$\mathcal{T}^{\mathrm{n.o}}_{\bar\rho,\Sigma}(w, w_0) \underset{\Lambda^{\mathrm{n.o}}}{\otimes} \Lambda^{\mathrm{n.o}}/P^{\mathrm{n.o}}_{\alpha} \simeq \mathcal{T}^{\mathrm{n.o}}_{\alpha}(w, w_0),$$

where $\mathcal{T}^{\mathrm{n.o}}_{\alpha}(w, w_0)$ denotes the Hecke algebra acting on $e\,\mathrm{H}^d(Y_{11}(p^\alpha), \mathcal{L}(w, w_0; \mathcal{O}))$.

4.5. **Variants.** The algebraic homomorphism

$$\Lambda^{\det} \to \mathcal{O},\ [u] \mapsto u^{(w_0 t - w)/2}$$

yields a surjective homomorphism $\Lambda^{\mathrm{n.o}} = \Lambda^{\det}\widehat{\otimes}\Lambda^{\mathrm{ord}} \to \Lambda^{\mathrm{ord}}$ used implicitly in the following definition:

$$\mathcal{R}^{\mathrm{ord}}_{\bar\rho,\Sigma}((w_0 t - w)/2) = \mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma} \otimes_{\Lambda^{\mathrm{n.o}}} \Lambda^{\mathrm{ord}}. \tag{19}$$

The Λ^{ord}-algebra $\mathcal{R}^{\mathrm{ord}}_{\bar\rho,\Sigma}((w_0 t - w)/2)$ parametrizes ordinary deformations of slope $(w_0 t - w)/2$ (see [H1]). In parallel weight ($w_\tau = w_0$ for all τ) the corresponding homomorphism $\Lambda^{\det} \to \mathcal{O}$ is the trivial one. The ring $\mathcal{R}^{\mathrm{ord}}_{\bar\rho,\Sigma} = \mathcal{R}^{\mathrm{ord}}_{\bar\rho,\Sigma}(0)$ then parametrizes deformations which are locally reducible at all places dividing p with unramified one dimensional quotients. By adapting the proof of theorem 4.6 we obtain:

Corollary 4.8. *Assume ($\star$) and ($\star\star$) with an ordinary $\bar\pi$. Then $\mathcal{H}^{\mathrm{ord}}_{\bar\rho,\Sigma}(\bar w, \bar w_0)$ is free of rank one over $\mathcal{T}^{\mathrm{ord}}_{\bar\rho,\Sigma}(\bar w, \bar w_0)$ and the natural surjection $\mathcal{R}^{\mathrm{ord}}_{\bar\rho,\Sigma}((\bar w_0 t - \bar w)/2) \twoheadrightarrow \mathcal{T}^{\mathrm{ord}}_{\bar\rho,\Sigma}(\bar w, \bar w_0)$ is an isomorphism of algebras which are finite flat of complete intersection over Λ^{ord}.*

If $\bar w = \bar w_0 t$ then for every $w_0 \geq 0$ the module $\mathcal{H}^{\mathrm{ord}}_{\bar\rho,\Sigma}(w_0 t, w_0)$ is free of rank one over $\mathcal{T}^{\mathrm{ord}}_{\bar\rho,\Sigma}(w_0 t, w_0)$ and the natural surjection $\mathcal{R}^{\mathrm{ord}}_{\bar\rho,\Sigma} \twoheadrightarrow \mathcal{T}^{\mathrm{ord}}_{\bar\rho,\Sigma}(w_0 t, w_0)$ is an isomorphism of algebras which are finite flat of complete intersection over Λ^{ord}.

The algebraic homomorphism $\Lambda^{\mathrm{ord}} \to \mathcal{O}$ induced by

$$\det(\rho_{\bar\pi,p})\widetilde{\det(\bar\rho)}^{-1}\chi\omega^{-1} : \mathrm{Gal}_{F,p\Sigma} \to \mathcal{O}^\times$$

yields a surjective homomorphism $\Lambda^{\mathrm{n.o}} = \Lambda^{\mathrm{ord}}\widehat{\otimes}\Lambda^{\det} \to \Lambda^{\det}$ used implicitly in the following definition:

$$\mathcal{R}^{\det}_{\bar\rho,\Sigma} = \mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma} \otimes_{\Lambda^{\mathrm{n.o}}} \Lambda^{\det} \tag{20}$$

The $\Lambda^{\det}$-algebra $\mathcal{R}^{\det}_{\bar\rho,\Sigma}$ parametrizes nearly ordinary deformations of $\bar\rho$ with determinant equal to $\det(\rho_{\bar\pi,p})$. Again, by adapting the proof of theorem 4.6 we obtain:

Corollary 4.9. *Under ($\star$) and ($\star\star$), the module $\mathcal{H}^{\det}_{\bar\rho,\Sigma}(w, \bar w_0)$ is free of rank one over $\mathcal{T}^{\det}_{\bar\rho,\Sigma}(w, \bar w_0)$ and the natural surjection $\mathcal{R}^{\det}_{\bar\rho,\Sigma} \twoheadrightarrow \mathcal{T}^{\det}_{\bar\rho,\Sigma}(w, \bar w_0)$ is an isomorphism of algebras which are finite flat of complete intersection over $\Lambda^{\det}$.*

5. Analytic p-adic L-function for Hida families

In this last section of the article we will apply the results obtained in the previous sections to the construction of analytic p-adic L-functions for Hida families of Hilbert automorphic forms. Partial results in the ordinary case can be found in [DO] where Kitagawa's classical approach has been generalized. A different rather technical approach using p-adic Rankin-Selberg convolution has been used by [M1] in the ordinary case, and is also expected to work in the nearly ordinary case along the lines of [H4].

5.1. **Universal nearly ordinary p-adic automorphic symbol.** In this section we will describe a p-adic limit interpolation procedure that allows to extract the lowest coefficient of the local system $\mathcal{L}_K(w, w_0; E/\mathcal{O})$ even if it is not critical.

For β and γ large enough so that K_p contains $M(p^\beta, p^{\beta+\gamma})$, consider the following map defined in §1.3.2

$$C^{\Sigma,\gamma}_{K,\beta} = C_K(p^\beta\Sigma, p^{\beta+\gamma}\Sigma\,\mathfrak{a}_K) : \mathbb{A}^\times / F^\times U(p^{\beta+\gamma}\Sigma\,\mathfrak{a}_K) \to Y_K.$$

Since for all $u \in U(p^{\beta+\gamma}\Sigma\,\mathfrak{a}_K)$ one has $\left(\begin{smallmatrix} u_p & (u_p-1)p^{-\beta} \\ 0 & 1\end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right) \pmod{p^\gamma}$ one deduces that the sheaf $C^{\Sigma,\gamma *}_{K,\beta}\mathcal{L}_K(w, w_0; p^{-\gamma}/\mathcal{O})$ is trivial, yielding an evaluation map on cohomology:

$$S^{\Sigma,\gamma}_{K,\beta}(w, w_0) : \mathrm{H}^d_c(Y_K, \mathcal{L}_K(w, w_0; p^{-\gamma}/\mathcal{O})) \to \mathcal{L}(w, w_0; p^{-\gamma}/\mathcal{O})[[\mathrm{Cl}^+_F(p^{\beta+\gamma}\Sigma\,\mathfrak{a}_K)]].$$

By lemma 1.12 and the fact that $p^{\frac{w-w_0t}{2}}\left(\begin{smallmatrix} p & \star \\ 0 & 1\end{smallmatrix}\right)$ acts trivially on the lowest weight vector $Y^w \in \mathcal{L}_K(w, w_0; p^{-\gamma}/\mathcal{O})$ follows the commutativity of the diagram:

(21)

$$\begin{tikzcd}
\mathrm{H}^d_c(Y_K, \mathcal{L}_K(w, w_0; p^{-\gamma}/\mathcal{O})) \arrow[r, "p^{\frac{w-w_0t}{2}} U_p = U_p^0"] \arrow[d, "S^{\Sigma,\gamma}_{K,\beta+1}(w{,}w_0)"] & \mathrm{H}^d_c(Y_K, \mathcal{L}_K(w, w_0; p^{-\gamma}/\mathcal{O})) \arrow[d, "S^{\Sigma,\gamma}_{K,\beta}(w{,}w_0)"] \\
\mathcal{L}(w, w_0; p^{-\gamma}/\mathcal{O})[[\mathrm{Cl}^+_F(p^{\beta+1+\gamma}\Sigma\,\mathfrak{a}_K)]] \arrow[d, "\mathrm{low}^{w,w_0}_{\beta+1}"] & \mathcal{L}(w, w_0; p^{-\gamma}/\mathcal{O})[[\mathrm{Cl}^+_F(p^{\beta+\gamma}\Sigma\,\mathfrak{a}_K)]] \arrow[d, "\mathrm{low}^{w,w_0}_{\beta}"] \\
(p^{-\gamma}/\mathcal{O})[\mathrm{Cl}^+_F(p^{\beta+1+\gamma}\Sigma\,\mathfrak{a}_K)] \arrow[r, "\mathrm{pr}^{\beta}_{\beta+1}"] & (p^{-\gamma}/\mathcal{O})[\mathrm{Cl}^+_F(p^{\beta+\gamma}\Sigma\,\mathfrak{a}_K)]
\end{tikzcd}$$

where $\mathrm{low}^{w,w_0}_\beta$ is the projection to the coefficient of the lowest weight vector. If follows that the maps $\mathrm{low}^{w,w_0}_\beta \circ S^{\Sigma,\gamma}_{K,\beta}(w, w_0) \circ (U^0_p)^{-\beta}$ form a projective system with respect to β and by passing to the limit we obtain:

$$S^{\Sigma,\gamma}_K(w, w_0) : e_p\,\mathrm{H}^d_c(Y_K, \mathcal{L}_K(w, w_0; p^{-\gamma}/\mathcal{O})) \to p^{-\gamma}/\mathcal{O}[[\mathrm{Cl}^+_F(p^\infty\Sigma\,\mathfrak{a}_K)]]. \tag{22}$$

It is clear that $(S^{\Sigma,\gamma}_K(w, w_0))_{K,\gamma}$ form an inductive system with respect to γ and the maps induced by the natural projections $Y_{K'} \to Y_K$, for $K' \subset K$. We are now in position to study the variation of p-adic automorphic symbol when K_p shrinks. Namely, under a mild restrictions on K allowing to ignore $\mathfrak{a}_K$, we have a homomorphism

$$\left(S^{\Sigma,\gamma}_K(w, w_0)\right)_{K,\gamma} : \varinjlim_K e_p\,\mathrm{H}^d_c(Y_K, \mathcal{L}_K(w, w_0; E/\mathcal{O})) \to E/\mathcal{O}[[\mathrm{Cl}^{(p)}_F(p^\infty\Sigma)]].$$

where Y_K runs over a given tower of Hilbert modular varieties.

5.2. **Analytic p-adic L-functions for ordinary families.** Let Σ be a finite set of primes outside p and $\bar{\rho} : \mathrm{Gal}_{F,p\Sigma} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a representation, satisfying $(\star)$ and $(\star\star)$ with ordinary $\bar{\pi}$ of parallel weight $(\bar{w} = \bar{w}_0 t)$.

5.2.1. *Construction.* Let $Y_1(p^\alpha)$ be the Hilbert modular variety of level $K_\alpha = K^p_{\bar{\rho},\Sigma} K_1(p^\alpha)$ (see §2.3.1). By lemma 2.8 we have $\mathrm{Cl}_F^{(p)}(p^\infty \Sigma\, \mathfrak{a}_K) = \mathrm{Cl}_F^{(p)}(p^\infty \Sigma)$.

For every $w \in \mathbb{N}$, the universal ordinary p-adic automorphic symbol $S_\Sigma^{\mathrm{ord}}(wt, w)$ is defined as the inductive limit over α and γ of the maps $(S_{K_\alpha}^{\Sigma,\gamma}(wt, w))_{\alpha,\gamma}$ from (22):

$$S_\Sigma^{\mathrm{ord}}(wt, w) : \varinjlim_{\alpha,\gamma} e_p \, \mathrm{H}_c^d(Y_1(p^\alpha), \mathcal{L}_K(wt, w; p^{-\gamma}/\,\mathcal{O})) \to E/\,\mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty \Sigma)]].$$

In the notations of §3.2 we have

$$S_\Sigma^{\mathrm{ord}}(wt, w) \in \mathcal{H}_1(wt, w) \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty \Sigma)]].$$

By corollary 4.8, $\mathcal{H}_{\bar{\rho},\Sigma}^{\mathrm{ord}} = \mathcal{H}_{\bar{\rho},\Sigma}^{\mathrm{ord}}(0,0)$ is free of rank one over $\mathcal{T}_{\bar{\rho},\Sigma}^{\mathrm{ord}} = \mathcal{T}_{\bar{\rho},\Sigma}^{\mathrm{ord}}(0,0)$. Denote by $b_{\bar{\rho},\Sigma}^{\mathrm{ord}}$ a basis. Let $S_{\bar{\rho},\Sigma}^{\mathrm{ord}}$ be the image of $S_\Sigma^{\mathrm{ord}}(0,0)$ in $\mathcal{H}_{\bar{\rho},\Sigma}^{\mathrm{ord}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty \Sigma)]]$.

Definition 5.1. We define the universal ordinary p-adic L-function

$$L_p^{\mathrm{ord}} = L_p^{\mathrm{ord}}(\bar{\rho}, \Sigma) \in \mathcal{T}_{\bar{\rho},\Sigma}^{\mathrm{ord}}[[\mathrm{Cl}_F^{(p)}(p^\infty \Sigma)]]$$

as the coordinate of $S_{\bar{\rho},\Sigma}^{\mathrm{ord}}$ in the basis $b_{\bar{\rho},\Sigma}^{\mathrm{ord}} \otimes 1$.

Remark 5.2. Although Λ^{ord} and $\mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty \Sigma)]]$ are abstractly isomorphic, the action of the former on $\mathcal{R}_{\bar{\rho},\Sigma}^{\mathrm{n.o}}$ is defined using the determinant, while the action of the latter is defined using twists.

5.2.2. *Dependence on the weight.* One of the main features of Hida's theory is that objects constructed in one given weight can be transfered to other weights. We will now show that L_p^{ord} has this feature too.

By theorem 3.1(ii) there is a natural isomorphism $\mathcal{H}_{\bar{\rho},\Sigma}^{\mathrm{ord}} \xrightarrow{\sim} \mathcal{H}_{\bar{\rho},\Sigma}^{\mathrm{ord}}(wt, w)$, compatible with the isomorphism of Hecke algebras

$$\mathcal{T}_{\bar{\rho},\Sigma}^{\mathrm{ord}} \xrightarrow{\sim} \mathcal{T}_{\bar{\rho},\Sigma}^{\mathrm{ord}}(wt, w), \tag{23}$$

itself compatible with the algebra automorphism of Λ^{ord} given by $[z] \mapsto \chi(z)^{-w} \omega(z)^w [z]$. The image $b_{\bar{\rho},\Sigma}^{\mathrm{ord}}(w)$ of $b_{\bar{\rho},\Sigma}^{\mathrm{ord}}$ under this isomorphism is clearly a basis of $\mathcal{H}_{\bar{\rho},\Sigma}^{\mathrm{ord}}(wt, w)$ over $\mathcal{T}_{\bar{\rho},\Sigma}^{\mathrm{ord}}(wt, w)$.

Denote by $S_{\bar{\rho},\Sigma}^{\mathrm{ord}}(wt, w)$ the image of $S_\Sigma^{\mathrm{ord}}(wt, w)$ in $\mathcal{H}_{\bar{\rho},\Sigma}^{\mathrm{ord}}(wt, w) \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty \Sigma)]]$ and define the p-adic L-function

$$L_p^{\mathrm{ord}}(w) = L_p^{\mathrm{ord},w}(\bar{\rho}, \Sigma) \in \mathcal{T}_{\bar{\rho},\Sigma}^{\mathrm{ord}}[[\mathrm{Cl}_F^{(p)}(p^\infty \Sigma)]]$$

as the coordinate of $S_{\bar{\rho},\Sigma}^{\mathrm{ord}}(wt, w)$ in the basis $b_{\bar{\rho},\Sigma}^{\mathrm{ord}}(w) \otimes 1$.

In vertu of the above choices of basis we have:

Lemma 5.3. *The natural isomorphism*

$$j_w : \mathcal{T}^{\mathrm{ord}}_{\bar\rho,\Sigma}\,\widehat{\otimes}_{\mathcal{O}}\,\mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)]] \xrightarrow{\sim} \mathcal{T}^{\mathrm{ord}}_{\bar\rho,\Sigma}(wt,w)\widehat{\otimes}_{\mathcal{O}}\,\mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)]]$$

induced by (23) sends L_p^{ord} to $L_p^{\mathrm{ord}}(w)$.

5.2.3. *Proof of theorem 0.3.* We are now is position to prove interpolation property in any parallel weight. Let π be a cohomological cuspidal automorphic representation of weight (wt,w), ordinary of level $K_1(p^\alpha)$ at p and such that $\rho_{\pi,p}$ is a deformation of $\bar\rho$. It defines an algebra homomorphism $\mathcal{T}^{\mathrm{ord}}_{\bar\rho,\Sigma}(wt,w)\longrightarrow\mathcal{O}$ extending naturally to a homomorphism:

$$\theta_\pi : \mathcal{T}^{\mathrm{ord}}_{\bar\rho,\Sigma}(wt,w)[[\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)]] \to \mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)]].$$

Since the constructions from §1.5 and §5.1 coincide in parallel (critical) weight (wt,w), we have $\theta_\pi(L_p^{\mathrm{ord}}(w)) = L_p^\Sigma(\pi)$, hence $\theta_\pi(j_w(L_p^{\mathrm{ord}})) = L_p^\Sigma(\pi)$ (see lemma 5.3).

Theorem 0.3 then follows from the fact that $\mathcal{T}^{\mathrm{ord}}_{\bar\rho,\Sigma}$ is isomorphic to $\mathcal{R}^{\mathrm{ord}}_{\bar\rho,\Sigma}$ as a Λ^{ord}-algebra (see corollary 4.8).

Using theorem 2.11 one can further specialize L_p^{ord} by a character of $\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)$ obtaining the following:

Corollary 5.4. *For any $w_0\in\mathbb{Z}$ such that $|w_0|\le w$ and for any finite order character ϕ of $\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)$, ramified at all the places dividing $p\Sigma$, the specialization of L_p^{ord} by $\Big([a]\mapsto\chi(a)^{\frac{w-w_0}{2}}\omega(a)^{-\frac{w-w_0}{2}}\phi(a)\Big)\circ\theta_\pi\circ j_w$ equals*

$$\frac{L^{(p\Sigma)}(\pi\otimes\phi\omega^{-\frac{w-w_0}{2}},\frac{w-w_0}{2}+1)\Gamma(\pi,\frac{w-w_0}{2}+1)}{\Omega^\Sigma_{\pi,\infty}}\tau(\phi,\xi)\prod_{v|p}\alpha_v^{-\,\mathrm{cond}(\phi_v)}.$$

Hence $L_p^{\mathrm{ord}} = L_p^{\mathrm{ord}}(\bar\rho,\Sigma)$ is rightfully called a p-adic L-function, since it is uniquely determined by p-adic interpolation of special values of *classical* L-functions, and is also rightfully called universal, since it can be specialized to the p-adic L-function associated to any ordinary cuspidal automorphic representation lifting $\bar\rho$.

5.3. **Analytic p-adic L-function for nearly ordinary families.** Let Σ be a finite set of primes outside p and $\bar\rho:\mathrm{Gal}_{F,p\Sigma}\to\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a representation satisfying $(\star)$ and $(\star\star)$. Since $\bar\rho$ is nearly ordinary and distinguished, by §4.1 there exists a universal nearly ordinary deformation ring $\mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma}$. Since p is odd, every deformation of $\bar\rho$ has a twist by a character of $\mathrm{Gal}^{\mathrm{ab},p-\mathrm{part}}_{F,p\Sigma}\simeq\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)$ having determinant $\det(\rho_{\bar\pi,p})$. Hence we have a canonical isomorphism:

$$\mathcal{R}^{\mathrm{n.o}}_{\bar\rho,\Sigma} = \mathcal{R}^{\mathrm{det}}_{\bar\rho,\Sigma}[[\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)]].$$

Let $Y_{11}^{\mathrm{ad}}(p^\alpha)$ denote the Hilbert modular variety of level $K_\alpha = K^p_{\bar\rho,\Sigma}K_{11}^{\mathrm{ad}}(p^\alpha)$ (see §2.3.1). By lemma 2.8 we have $\mathrm{Cl}_F^{(p)}(p^\infty\Sigma\,\mathfrak{a}_K) = \mathrm{Cl}_F^{(p)}(p^\infty\Sigma)$.

The universal nearly ordinary p-adic automorphic symbol $S_\Sigma^{\mathrm{n.o}}$ is defined as the inductive limit over α and γ of the maps $(S_{K_\alpha}^{\Sigma,\gamma}(\bar{w}_0 t, \bar{w}_0))_{\alpha,\gamma}$ from (22):

$$S_\Sigma^{\mathrm{n.o}} : \varinjlim_{\alpha,\gamma} e_p \, \mathrm{H}_c^d(Y_{11}^{\mathrm{ad}}(p^\alpha), \mathcal{L}_K(\bar{w}_0 t, \bar{w}_0; p^{-\gamma}/\,\mathcal{O})) \to E/\,\mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)]].$$

In the notations of §3.2 we have

$$S_\Sigma^{\mathrm{n.o}} \in \mathcal{H}_{\mathrm{ad}}(\bar{w}_0 t, \bar{w}_0) \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)]].$$

Denote by $S_{\bar\rho,\Sigma}^{\mathrm{n.o}}$ the image of $S_\Sigma^{\mathrm{n.o}}$ in $\mathcal{H}_{\bar\rho,\Sigma}^{\det}(\bar{w}_0 t, \bar{w}_0) \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)]]$.

By corollary 4.9, $\mathcal{H}_{\bar\rho,\Sigma}^{\det}(\bar{w}_0 t, \bar{w}_0)$ is free of rank one over $\mathcal{T}_{\bar\rho,\Sigma}^{\det}(\bar{w}_0 t, \bar{w}_0)$ with basis $b_{\bar\rho,\Sigma}^{\det}$ and $\mathcal{T}_{\bar\rho,\Sigma}^{\det}(\bar{w}_0 t, \bar{w}_0)$ is isomorphic to $\mathcal{R}_{\bar\rho,\Sigma}^{\det}$ as a $\Lambda^{\det}$-algebra.

Definition 5.5. We define the p-adic nearly ordinary L-function

$$L_p^{\mathrm{n.o}}(\bar\rho, \Sigma) \in \mathcal{T}_{\bar\rho,\Sigma}^{\det}(\bar{w}_0 t, \bar{w}_0)[[\mathrm{Cl}_F^{(p)}(p^\infty\Sigma)]] \simeq \mathcal{R}_{\bar\rho,\Sigma}^{\mathrm{n.o}},$$

as the coordinate of $S_{\bar\rho,\Sigma}^{\mathrm{n.o}}$ in the basis $b_{\bar\rho,\Sigma}^{\det} \otimes 1$.

Let π be a nearly ordinary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ of weight $(\bar{w}_0 t, \bar{w}_0)$ such that $\rho_{\pi,p}$ is a deformation of $\bar\rho$ with determinant $\det(\rho_{\bar\pi,p})$. It defines an algebra homomorphism

$$\theta_\pi : \mathcal{T}_{\bar\rho,\Sigma}^{\det}(\bar{w}_0 t, \bar{w}_0) \longrightarrow \mathcal{O}\,.$$

Since the constructions from §1.5 and §5.1 coincide in weight $(\bar{w}_0 t, \bar{w}_0)$ we have

$$\theta_\pi(L_p^{\mathrm{n.o}}(\bar\rho, \Sigma)) = L_p^\Sigma(\pi)$$

as claimed in theorem 0.2.

Hence $L_p^{\mathrm{n.o}} = L_p^{\mathrm{n.o}}(\bar\rho, \Sigma)$ is rightfully called a p-adic L-function, since it is uniquely determined by p-adic interpolation of special values of *classical* L-functions.

Contrary to the ordinary case, we do not know whether the specialization of $L_p^{\mathrm{n.o}}$ by an arbitrary algebraic homomorphism $\mathcal{R}_{\bar\rho,\Sigma}^{\det} \to \mathcal{O}$ yields a p-adic L-function associated to some nearly ordinary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. We believe that the resolution of this question should involve a new construction allowing to lift (non-canonically) these measures to measures in $2d$-variables, similarly to Shintani's and Deligne-Ribet's construction of the p-adic L function of a totally real number field. This is a subtle issue to which we hope to come back in a future work.

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