# $p$-ADIC $L$-FUNCTIONS FOR HILBERT MODULAR FORMS

MLADEN DIMITROV

The use of modular symbols to attach $p$-adic $L$-functions to Hecke eigenforms goes back to the work of Manin *et al* in the 70s. In the 90s Stevens proposed a new approach based on his theory of overconvergent modular symbols, which was successfully used to construct $p$-adic $L$-functions on the eigencurve for $\mathrm{GL}_2$ over $\mathbb{Q}$. Recently, building on Urban's construction of eigenvarieties for general reductive groups and on the author's theory of automorphic symbols for $\mathrm{GL}_2$ over a totally real number field, Barrera gave a new construction of $p$-adic $L$-functions for Hilbert modular forms using the overconvergent compactly supported cohomology of Hilbert modular varieties.

In addition to giving an overview of these topics, the lecture notes also contain some original results such as the precise correspondence between automorphic and modular symbols for $\mathrm{GL}_2$ over totally real number fields.

## 1. $p$-adic $L$-functions for elliptic modular forms

In this section we present the main steps in Stevens' construction of $p$-adic $L$-functions of elliptic modular forms, following [S], [PS] and [Be].

1.1. **Modular symbols.** The group $\mathrm{GL}_2(\mathbb{Q})$ acts on the left on $\mathbb{P}^1(\mathbb{Q})$ by linear fractional transformations, hence acts on the group of degree 0 divisors:

$$\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) = \left\{ \sum_{x\in\mathbb{P}^1(\mathbb{Q})} m_x x | m_x \in \mathbb{Z}, m_x = 0 \text{ for all but finitely many } x, \sum_{x\in\mathbb{P}^1(\mathbb{Q})} m_x = 0 \right\}.$$

For any congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and any right $\Gamma$-module $M$, $\Gamma$ acts on the right on $\mathrm{Hom}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), M)$ by:

$$(\phi_{|\gamma})(D) = \phi(\gamma \cdot D)_{|\gamma} \text{ for all } \gamma \in \Gamma, \phi \in \mathrm{Hom}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), M) \text{ and } D \in \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})).$$

**Definition 1.1.** The space of $M$-valued modular symbols on $\Gamma$ is defined as:

$$\mathrm{Symb}_\Gamma(M) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), M)^\Gamma = \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), M).$$

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It follows from the definition that for any commutative ring $R$ and any $R[\Gamma]$-module $M$, $\mathrm{Symb}_\Gamma(M)$ inherits an $R$-module structure. Moreover any flat ring homomorphism $R \to R'$ induces a natural isomorphism:

$$\mathrm{Symb}_\Gamma(M) \otimes_R R' \xrightarrow{\sim} \mathrm{Symb}_\Gamma(M \otimes_R R').$$

1.2. **Hecke action.** Suppose that $M$ is a right module over the monoid $\Delta$ generated by $\Gamma$ and some $x \in \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{M}_2(\mathbb{Z})$. The Hecke operator $[\Gamma x \Gamma]$ sends $\phi \in \mathrm{Symb}_\Gamma(M)$ to

$$\phi_{|[\Gamma x \Gamma]} = \sum_i \phi_{|x_i} \text{ , where } \Gamma x \Gamma = \coprod_i \Gamma x_i.$$

For $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$ and a prime $\ell \nmid N$, the Hecke operator $T_\ell$ is defined as:

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \Gamma = \Gamma \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \coprod_{a=0}^{\ell-1} \Gamma \begin{pmatrix} 1 & a \\ 0 & \ell \end{pmatrix} \coprod \Gamma \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

For $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$ and a prime $p \mid N$, the Hecke operator $U_p$ is defined as:

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma = \coprod_{a=0}^{p-1} \Gamma \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}. \tag{2}$$

**Remark 1.2.** For any $\phi \in \mathrm{Symb}_\Gamma(\mathbb{Z}) \simeq \mathrm{Hom}(\Gamma \setminus \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), \mathbb{Z})$ one has:

$$(\phi_{|U_p})(\Gamma(\infty - 0)) = \sum_{a=0}^{p-1} \phi(\Gamma(\infty - \tfrac{a}{p})).$$

1.3. **Duals.** Given any right $R[\Gamma]$-module $M$, $\Gamma$ acts on the right on the dual module $M^\vee = \mathrm{Hom}_R(M, R)$ by letting $\lambda_{|\gamma}(m) = \lambda(m_{|\gamma^{-1}})$ and the canonical pairing

$$M \times M^\vee \to R, \quad (m, \lambda) \mapsto \lambda(m)$$

is $\Gamma$-equivariant.

Assume that $\Gamma$ is preserved by the anti-involution

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \gamma^* = \det(\gamma)\gamma^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right). \tag{3}$$

Then any right $R[\Gamma]$-module $M$, can be seen as a left $R[\Gamma]$-module by letting:

$$\gamma \cdot m = m_{|\gamma^*}.$$

and *vice-versa*. In particular the left $R[\Gamma]$-module $M^* = \mathrm{Hom}_R(M, R)$ for the action $(\gamma \cdot \lambda)(m) = \lambda(m_{|\gamma})$ can be viewed as right $R[\Gamma]$-module via $(\lambda * \gamma)(m) = \lambda(m_{|\gamma^*})$. If we assume further that $M$ has a central character $\omega_M$, then one has an isomorphism of right $R[\Gamma]$-modules:

$$M^* \simeq M^\vee \otimes \omega_M. \tag{4}$$

1.4. **Sheaf cohomology.** Assume that $\Gamma$ is torsion free (for example $\Gamma \subset \Gamma_1(N)$ with $N > 3$). Then $\Gamma$ acts freely (by linear fractional transformations) on the upper half-plane $\mathcal{H}$ in $\mathbb{C}$, and the modular curve $Y_\Gamma = \Gamma \backslash \mathcal{H}$ admits $\mathcal{H}$ as a covering space.

Given a right $\Gamma$-module $M$, we consider the local system:

$$\Gamma \backslash (\mathcal{H} \times M) \to Y_\Gamma$$

with left $\Gamma$-action given by $\gamma \cdot (z, m) = (\gamma \cdot z, m_{|\gamma^*})$. Denote by $\mathcal{M}$ the sheaf of locally constant sections, where $M$ is endowed with the discrete topology. It is well known that for $* \in \{\varnothing, c\}$ one has a natural Hecke equivariant isomorphism:

$$\mathrm{H}^\bullet_*(\Gamma, M) \xrightarrow{\sim} \mathrm{H}^\bullet_*(Y_\Gamma, \mathcal{M}).$$

The following result is due to Ash and Stevens (see [AS]):

**Theorem 1.3.** *There exists a Hecke equivariant isomorphism* $\iota_\Gamma : \mathrm{Symb}_\Gamma(M) \xrightarrow{\sim} \mathrm{H}^1_c(\Gamma, \mathcal{M})$.

1.5. **Symbols for modular forms.** For $k \geq 0$, we let $V_k(R)$ denote the ring of polynomials of degree at most $k$ over a commutative ring $R$. If we set for $\gamma \in \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{M}_2(\mathbb{Z})$ and $P \in V_k(R)$

$$P_{|\gamma}(z) = (cz + d)^k P\left(\frac{az + b}{cz + d}\right). \tag{5}$$

we obtain a right action of $\mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{M}_2(\mathbb{Z})$ on $V_k(R)$. By the discussion in §1.3, $V_k^*(R)$ has also a right action of $\gamma \in \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{M}_2(\mathbb{Z})$ sending $\ell \in V_k^*(R)$ to

$$(\ell_{|\gamma})(P) = \ell(P_{|\gamma^*}) = \ell\left((a - cz)^k P\left(\frac{dz - b}{a - cz}\right)\right). \tag{6}$$

Let $S_{k+2}(\Gamma)$ denote as usual the complex space of cuspforms of weight $k+2$ and level $\Gamma$. For any $f \in S_{k+2}(\Gamma)$ and any $D \in \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ we consider $\phi_f(D) \in V_k^*(\mathbb{R})$ such that

$$\phi_f(D)(P) = \mathrm{Re}\left(\int_D f(z)P(z)dz\right) \text{ for all } P \in V_k(\mathbb{R}).$$

A direct computation shows that $\phi_f \in \mathrm{Symb}_\Gamma(V_k^*(\mathbb{R}))$.

**Theorem 1.4.** *There exists a commutative diagram:*

$$\begin{tikzcd} S_{k+2}(\Gamma) \arrow[r, "\delta_\Gamma", "\sim"'] \arrow[d, hook, "\phi"'] & \mathrm{H}^1_!(\Gamma, V_k^*(\mathbb{R})) \\ \mathrm{Symb}_\Gamma(V_k^*(\mathbb{R})) \arrow[r, "\iota_\Gamma", "\sim"'] & \mathrm{H}^1_c(\Gamma, V_k^*(\mathbb{R})) \arrow[u, two heads] \end{tikzcd},$$

*where* $\delta_\Gamma$ *is the Eichler-Shimura isomorphism.*

Assume in the sequel that $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ normalises $\Gamma$. Then $Y_\Gamma$ is defined over $\mathbb{R}$ and the Hecke operator

$$T_\infty = \Gamma\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Gamma = \Gamma\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

commutes with those introduced in (1) and (2).

For any $f \in S_{k+2}(\Gamma)$ and any $D \in \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ define $\phi_f^{\pm}(D) \in V_k^*(\mathbb{C})$ by

$$\phi_f^{\pm}(D)(P) = \int_D f(z)P(z)dz \pm \int_{-D} f(z)P(-z)dz, \text{ for all } P \in V_k(\mathbb{C}). \tag{7}$$

**Theorem 1.5.** *There exists a commutative diagram:*

$$\begin{tikzcd} S_{k+2}(\Gamma) \arrow[r, "\delta_\Gamma^{\pm}", "\sim"'] \arrow[d, hook, "\phi^{\pm}"'] & \mathrm{H}^1_!(\Gamma, V_k^*(\mathbb{C}))^{\pm} \\ \mathrm{Symb}_\Gamma^{\pm}(V_k^*(\mathbb{C})) \arrow[r, "\iota_\Gamma", "\sim"'] & \mathrm{H}^1_c(\Gamma, V_k^*(\mathbb{C}))^{\pm} \arrow[u, two heads] \end{tikzcd},$$

*where $\pm$ denotes the subspace on which $T_\infty = \pm 1$.*

1.6. **Complex $L$-functions.** The complex $L$-function of $f(z) = \sum_{n\geq 1} a_n e^{2i\pi nz} \in S_{k+2}(\Gamma)$ is defined for $\mathrm{Re}(s) > (k+3)/2$ by the absolutely convergent Dirichlet series:

$$L(f,s) = \sum_{n\geq 1} \frac{a_n}{n^s},$$

which admits an analytic continuation to the entire complex plane and satisfies a functional equation relating $s$ to $k+2-s$.

More generally, given any Dirichlet character $\chi$ we define the imprimitive $L$-function of $f$ twisted by $\chi$ as:

$$L(f,\chi,s) = \sum_{n\geq 1} \frac{a_n\chi(n)}{n^s}.$$

The main ingredient in computing special values of $L$-functions via modular symbols is the Mellin transform formula which states that in the domain of absolute convergence:

$$L(f,s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f(iy)y^{s-1}dy. \tag{8}$$

Another important ingredient is the following result, known under the name of Birch's lemma, allowing to compute twisted $L$-values using modular symbols (see [MTT]).

**Lemma 1.6.** *If $\chi$ is a primitive Dirichlet character of conductor $m$, then $L(f,\chi,s) = L(f_\chi, s)$ where*

$$f_{\bar\chi} = \frac{1}{\tau(\chi)} \sum_{a \bmod m} \chi(a) f(z + \tfrac{a}{m}), \text{ and } \tau(\chi) = \sum_{a \bmod m} \chi(a) e^{2i\pi a/m}.$$

Assume now that $f$ is a newform of level $N$, that is a normalised primitive eigenform for all the Hecke operators, and denote by $K_f$ the Hecke field $\mathbb{Q}(a_n, n \geq 1)$. By Theorem 1.5 $\phi_f^{\pm}$ is a non-zero vector of the complex line $\mathrm{Symb}^{\pm}_{\Gamma_1(N)}(V_k^*(\mathbb{C}))[f]$, where $[f]$ denotes the subspace on which the Hecke operators act by the same eigenvalues as on $f$. It follows that there exists a period $\Omega_f^{\pm} \in \mathbb{C}^\times$ which is uniquely determined up to multiplication by an element of $K_f^\times$ and such that

$$\mathrm{Symb}^{\pm}_{\Gamma_1(N)}(V_k^*(K_f))[f] = K_f \cdot \phi_f^{\pm}/\Omega_f^{\pm}. \tag{9}$$

The following result, due to Manin, establishes the rationality of the critical values of $L(f, \chi, s)$ and is a prerequisite for attaching a $p$-adic $L$-function to $f$ via interpolation.

**Theorem 1.7.** *For any $0 \leq j \leq k$ and for any Dirichlet character $\chi$ one has*

$$\frac{L(f, \chi, j+1)}{\tau(\chi)\Omega_f^{\pm}(i\pi)^{j+1}} \in K_f(\chi), \text{ where } \pm = (-1)^j\chi(-1).$$

1.7. **Distributions.** We fix, once and for all, an embedding $\iota_p : \bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$. Denote by $v_p$ the unique valuation on $\bar{\mathbb{Q}}_p$ that extends the $p$-adic valuation on $\mathbb{Q}_p$, and we denote by $|\cdot|_p$ the corresponding norm.

Let $L$ be a finite extension of $\mathbb{Q}_p$ and choose an open compact subset $X$ of $\mathbb{Q}_p^r$.

We consider the space $A(X, L) = \varinjlim A_n(X, L)$ of locally $L$-analytic functions on $X$. By definition $f \in A_n(X, L)$ if for each $a \in X$ there exist coefficients $c_m(a) \in L$ indexed by $m \in \mathbb{N}^r$ such that

$$f(x) = \sum_{m \in \mathbb{N}^r} c_m(a)(x-a)^m, \text{ for all } x \in X, |x-a|_p < p^{-n}.$$

For each integer $n \geq 1$, $A_n(X, L)$ is a $L$-Banach space for the norm:

$$||f||_n = \sup_{a \in X, m \in \mathbb{N}^r} \left( |c_m(a)|_p p^{-n \sum_{i=1}^r m_i} \right).$$

The natural inclusion $A_n(X, L) \subset A_{n+1}(X, L)$ is compact, hence completely continuous (with dense image, since polynomials are dense).

The continuous linear $L$-dual $D_n(X, L)$ of $A_n(X, L)$ is a $L$-Banach space for the norm:

$$||\mu||_n = \sup_{f \in A_n(X,L)} \frac{|\mu(f)|_p}{||f||_n}.$$

The natural restriction maps $D_{n+1}(X, L) \subset D_n(X, L)$ are injective and compact, hence $D(X, L) = \varprojlim D_n(X, L)$ is a compact Frechet $L$-vector space, endowed with a family of norms $||\mu||_n = ||\mu_{|A_n(X,L)}||_n$.

**Definition 1.8.** The Frechet $D(X, L)$ is the space of $L$-valued *distributions* on $X$.

1.8. **Admissibility of a distribution.**

**Definition 1.9.** Let $h \in \mathbb{Q}_{\geq 0}$. A distributions $\mu \in D(X, L)$ is called *h-admissible* if there exists $C > 0$ such that $||\mu||_n \leq C \cdot p^{nh}$, for all $n \geq 1$. A 0-admissible (*i.e.* bounded) distribution is called a *measure*.

**Theorem 1.10** (Amice-Vélu [AV], Višik [V])**.** *For any $h \in \mathbb{N}$, an $h$-admissible distribution $\mu \in D(\mathbb{Z}_p, L)$ is uniquely determined by $\mu(\mathbb{1}_{a+p^n\mathbb{Z}_p} z^j)$ where $0 \leq j \leq h$, $n \in \mathbb{N}$ and $a \in \mathbb{Z}_p$.*

1.9. **Slope decomposition.** Suppose given an $L$-Banach space $V$ and a completely continuous endomorphism $u$ of $V$.

A classical result of Serre asserts that for any polynomial $Q \in L[T]$ there exists a $u$-stable direct sum decomposition $V = V_Q \oplus V'_Q$, with $V_Q$ finite dimensional, such that $Q(u)$ is nilpotent (resp. invertible) on $V_Q$ (resp. on $V'_Q$). This is called the Riesz decomposition and has been extended by Stevens and Urban (see [U]) to compact Frechet spaces.

**Definition 1.11.** For $h \in \mathbb{Q}_{\geq 0}$ and $V$ as above, we let $V^{<h}$ be the sum of $V_Q$ when $Q$ runs over polynomials whose roots in $\bar{\mathbb{Q}}_p$ have all valuation $< h$.

The space $V^{<h}$ is a finite dimensional $L$-vector space.

1.10. **Overconvergent cohomology.** Let $T$ be the standard diagonal torus of $\mathrm{GL}_2$ and denote by $B$ (resp. $\bar{B}$) the standard Borel (resp. the opposite Borel) containing $T$. Let $I = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$ be the standard Iwahori subgroup on $\mathrm{GL}_2(\mathbb{Z}_p)$.

Any continuous character $\lambda : T(\mathbb{Z}_p) \to L^\times$ can be extended to a character of $\bar{B}(\mathbb{Q}_p) \cap I$ by making the unipotent radical of $\bar{B}(\mathbb{Q}_p)$ act trivially. Consider space

$$A_\lambda(L) = \left\{ f : I \to L \text{ locally analytic and } f(bg) = \lambda(b)f(g), \forall b \in \bar{B}(\mathbb{Q}_p) \cap I \right\}.$$

Restriction to the unipotent radical $\left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ of $B(\mathbb{Z}_p)$ induces an isomorphism between $A_\lambda(L)$ and $A(\mathbb{Z}_p, L)$.

In the sequel we assume that $\lambda \left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right) = a^k$ for some $k \in \mathbb{N}$. The left action of $I$ on $A_k(L)$ (by right translation of the argument) corresponds to the following action on $A(\mathbb{Z}_p, L)$:

$$\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot f\right)(z) = (a - cz)^k f\left(\frac{dz - b}{a - cz}\right), \tag{10}$$

and extends by the same formula to an action of the monoid

$$\Delta = \mathrm{GL}_2(\mathbb{Q}_p) \cap \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

Note that $\Delta$ is generated as a monoid by $I$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. Denote by $A_k$ the space $A(\mathbb{Z}_p, L)$ endowed with the left action (10). The right action of $\gamma \in \Delta$ on its dual, which we denote by $D_k$, then sends $\mu \in D(\mathbb{Z}_p, L)$ to $\mu_{|\gamma}$ such that $\mu_{|\gamma}(f) = \mu(\gamma \cdot f)$.

**Lemma 1.12.** *The element* $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \in \Delta$ *sends* $f \in A_{k,n}$ *to* $f(\cdot p) \in A_{k,n-1}$ *and induces a compact operator on* $D_k$.

The natural restriction map:

$$D_k \to V_k^*(L), \quad \mu \mapsto \mu_{|V_k(L)} \tag{11}$$

is $\Delta$-equivariant for the respective actions defined here above and in (6).

Stevens shows that one has an exact sequence

$$0 \to D_{-2-k}(k+1) \to D_k \to V_k^*(L) \to 0$$

where $(k+1)$ denotes the twist with the $(k+1)$-th power of the determinant, and uses it to establish the following crucial for his construction of $p$-adic $L$-functions result.

**Theorem 1.13** (Stevens [S])**.** *For any* $k \in \mathbb{N}$ *the map*

$$\mathrm{Symb}_\Gamma^{\pm}(D_k)^{<k+1} \to \mathrm{Symb}_\Gamma^{\pm}(V_k^*(L))^{<k+1}$$

*induced by* (11) *is an isomorphism.*

1.11. **$p$-adic $L$-functions.** Recall that a $p$-stabilised newform is a normalised eigenform having the same eigenvalues as a given newform for all Hecke operators outside $p$, and which is in addition an eigenvector for $U_p$. Any newform of level $N$ divisible by $p$ is a $p$-stabilised newform itself. All other $p$-stabilised newforms $f$ have level $N$ exactly divisible by $p$ and are constructed as follows. One starts with a newform $g$ of level $N/p$ and for any root $\alpha$ of $X^2 - a_pX + \varepsilon(p)p^{k+1}$ one considers $f(z) = g(z) - \varepsilon(p)p^{k+1}\alpha^{-1}f(pz)$.

In the sequel we fix a $p$-stabilised newform $f \in S_{k+2}(\Gamma_1(N))$ whose $U_p$-eigenvalue $\alpha$ has valuation $h < k+1$ (this is referred to as the non-critical slope condition). Note that this implies in particular that $\alpha \neq 0$. By (9) one has elements

$$\phi_f^{\pm}/\Omega_f^{\pm} \in \mathrm{Symb}_{\Gamma_1(N)}^{\pm}(V_k^*(L))^{<k+1}.$$

and by Theorem 1.13 there exists a unique $\Phi_f^{\pm} \in \mathrm{Symb}_{\Gamma_1(N)}^{\pm}(D_k)^{<k+1}$ mapping to $\phi_f^{\pm}/\Omega_f^{\pm}$.

**Definition 1.14.** The $p$-adic $L$-function $L_p^{\pm}(f)$ of $f$ is defined as the restriction of the distribution $\Phi_f^{\pm}(\infty - 0)$ to $\mathbb{Z}_p^\times$.

**Theorem 1.15** (Stevens)**.** *The distribution* $L_p^{\pm}(f)$ *is $h$-admissible. Moreover it is uniquely determined by the following interpolation formula: for all* $0 \le j \le k$ *and for all Dirichlet characters* $\chi : \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}_p^\times$ *of conductor* $p^m$ *one has*

$$L_p^{\pm}(f, z^j\chi) = \iota_p\left(Z_p \cdot \frac{p^{m(j+1)}j!}{(-2i\pi)^j\tau(\bar{\chi})} \cdot \frac{L(f \otimes \bar{\chi}, j+1)}{\Omega_f^{\pm}}\right),$$

*where* $\pm = (-1)^j\chi(-1)$ *and* $Z_p = \frac{1}{\alpha^m}(1 - \frac{\varepsilon(p)p^{k-j}}{\alpha})(1 - \frac{p^j}{\alpha})$.

## 2. Cycles on Hilbert modular varieties

In this section we will recall the definition of automorphic cycles on Hilbert modular varieties introduced in [D1] and relate those to the modular cycles considered earlier by Manin [M] and Oda [O].

2.1. **Notations.** Let $F$ be a totally real number field of degree $d$, ring of integers $\mathfrak{o}$ and denote by $\Sigma$ be the set of its infinite places.

For any finite set of places $S$, we denote by $\mathbb{A}^{(S)}$ (resp. $\mathbb{A}_S$) the topological ring of adeles of $\mathbb{Q}$ outside $S$ (resp. at $S$). Let $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F = \mathbb{A}_F^{(\infty)} \times F_\infty$ be the ring of adeles of $F$.

We denote by $\widehat{\mathbb{Z}} = \prod_\ell \mathbb{Z}_\ell$ the profinite completion of $\mathbb{Z}$ and for any fractional ideal $\mathfrak{c}$ of $F$ we put $\widehat{\mathfrak{c}} = \widehat{\mathbb{Z}} \otimes \mathfrak{c}$.

We let $\mathfrak{d}$ denote the different of $F$, and for any fractional ideal $\mathfrak{c}$ of $F$ we let $\mathfrak{c}^* = \mathfrak{c}^{-1}\mathfrak{d}^{-1}$. Further we denote by $\mathfrak{c}_+ = \mathfrak{c} \cap F_+^\times$ the cone of totally positive elements of $\mathfrak{c}$. The narrow class group $\mathcal{C}\ell_F^+$ of $F$, which is the set of equivalence classes of $\mathfrak{c}$ modulo the action of $F_+^\times$, can be naturally identified with the strict idele class group $F^\times \setminus \mathbb{A}_F^\times / \widehat{\mathfrak{o}}^\times F_\infty^+$, where $F_\infty^+$ denotes the connected component of identity in $F_\infty^\times$. Fix a set of representatives $\mathfrak{c}_i$, $1 \leq i \leq h$, of $\mathcal{C}\ell_F^+$ and for each $i$ let $\eta_i \in \mathbb{A}_F^{(\infty)\times}$ be an idele generating $\mathfrak{c}_i^*$, *i.e.* $\mathfrak{c}_i = F \cap \eta_i \widehat{\mathfrak{o}} F_\infty$.

If $H$ is an algebraic group over $\mathbb{Q}$ and $S$ a finite set of places of $\mathbb{Q}$, the two natural projections induce an isomorphism:

$$H(\mathbb{A}) \xrightarrow{\sim} H(\mathbb{A}_S) \times H(\mathbb{A}^{(S)}), \quad h \mapsto (h_S, h^{(S)}).$$

By a slight abuse of notation we will also denote $h_S$ (resp. $h^{(S)}$) the element $(h_S, e^{(S)})$ (resp. $(e_S, h^{(S)})$) of $H(\mathbb{A})$, where $e$ denotes the identity element of $H(\mathbb{A})$.

The mirabolic group $M$ is defined as the semi-direct product $\mathbb{G}_m \ltimes \mathbb{G}_a$, where $\mathbb{G}_m$ acts on $\mathbb{G}_a$ by multiplication. We denote by $s : M \to \mathrm{GL}_2$ the natural inclusion sending $(y, x)$ to $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$.

Given an integral ideal $\mathfrak{f}$ of $F$ we let $M(\mathfrak{f}) = U(\mathfrak{f}) \ltimes \widehat{\mathfrak{o}}$, where $U(\mathfrak{f})$ consists of elements in $\widehat{\mathfrak{o}}^\times$ which are congruent to 1 modulo $\mathfrak{f}$. Denote by $E(\mathfrak{f})$ the subgroup of $\mathfrak{o}_+^\times$ of elements congruent to 1 modulo $\mathfrak{f}$, *i.e.* $E(\mathfrak{f}) = F^\times \cap U(\mathfrak{f}) F_\infty^+$.

2.2. **Hilbert modular varieties.** Let $G_\infty^+$ denote the connected component of identity in $\mathrm{GL}_2(F_\infty)$. The group $G_\infty^+$ acts transitively by linear fractional transformations on the unbounded hermitian symmetric domain $\mathfrak{H}_F = F_\infty \oplus F_\infty^+ \underline{i} \subset F \otimes \mathbb{C}$ where $\underline{i} = 1 \otimes \sqrt{-1}$. We have $\mathfrak{H}_F \simeq \mathfrak{H}^\Sigma$, where $\mathfrak{H}$ is Poincare's upper half-plane, the isomorphism being given by $\xi \otimes z \mapsto (\sigma(\xi)z)_{\sigma \in \Sigma}$, for $\xi \in F$ and $z \in \mathbb{C}$. The stabiliser $K_\infty^+$ of $\underline{i}$ in $G_\infty^+$ is the product of its center by its standard maximal compact subgroup, and there is an isomorphism:

$$G_\infty^+ / K_\infty^+ \xrightarrow{\sim} \mathfrak{H}_F, \quad g_\infty \mapsto g_\infty \cdot \underline{i}.$$

For an open compact subgroup $K$ of $\mathrm{GL}_2(\mathbb{A}_F^{(\infty)})$, the adelic Hilbert modular variety of level $K$ is defined as the locally symmetric space

$$Y_K := \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/KK_\infty^+ = \mathrm{GL}_2^+(F)\backslash(\mathfrak{H}_F \times \mathrm{GL}_2(\mathbb{A}_F^{(\infty)})/K),$$

where $\mathrm{GL}_2^+(F)$ denotes the subgroup of $\mathrm{GL}_2(F)$ of elements with determinant in $F_+^\times$.

Given a level $\mathfrak{n}$, an integral ideal of $\mathfrak{o}$, we consider the open compact subgroup:

$$K_1(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathfrak{o}}) | c \in \widehat{\mathfrak{n}}, d-1 \in \widehat{\mathfrak{n}} \right\},$$

and denote by $Y_1(\mathfrak{n})$ the corresponding Hilbert modular variety.

By Strong Approximation Theorem for $\mathrm{SL}_2/F$, the fibres of the map:

$$\det : Y_1(\mathfrak{n}) \to F^\times \backslash \mathbb{A}_F^\times / \ \widehat{\mathfrak{o}}^\times F_\infty^+,$$

are connected, hence $\pi_0(Y_1(\mathfrak{n})) \simeq \mathcal{C}\ell_F^+$.

For $1 \leq i \leq h$ the connected component $Y_1(\mathfrak{c}_i, \mathfrak{n}) = \det^{-1}(F^\times \eta_i \widehat{\mathfrak{o}}^\times F_\infty^+)$ is classically described as a quotient of $\mathfrak{H}_F$ by the congruence subgroup

$$\Gamma(\mathfrak{c}_i, \mathfrak{n}) = \mathrm{GL}_2(F) \cap \left(\begin{smallmatrix} \eta_i & 0 \\ 0 & 1 \end{smallmatrix}\right) K_1(\mathfrak{n}) \begin{pmatrix} \eta_i^{-1} & 0 \\ 0 & 1 \end{pmatrix} G_\infty^+ = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \begin{pmatrix} \mathfrak{o} & \mathfrak{c}_i^* \\ \mathfrak{c}_i \mathfrak{d} \mathfrak{n} & 1+\mathfrak{n} \end{pmatrix} \ \middle| \ ad - bc \in \mathfrak{o}_+^\times \right\}.$$

More precisely, there is an isomorphism:

$$\Gamma(\mathfrak{c}_i, \mathfrak{n})\backslash\mathfrak{H}_F \to Y_1(\mathfrak{c}_i, \mathfrak{n}), \quad x_\infty + y_\infty \underline{i} \mapsto \mathrm{GL}_2(F) \left(\begin{smallmatrix} y_\infty \eta_i & x_\infty \\ 0 & 1 \end{smallmatrix}\right) KK_\infty^+. \tag{12}$$

In general $Y_1(\mathfrak{c}_i, \mathfrak{n})$ is only a complex orbifold. In the sequel we assume that $\mathfrak{n}$ is *sufficiently divisible* in the sense of [D2, Lemma 2.1(iii)]. Then, for all $1 \leq i \leq h$, the group $\Gamma(\mathfrak{c}_i, \mathfrak{n})/(\Gamma(\mathfrak{c}_i, \mathfrak{n}) \cap F^\times)$ is torsion free, implying than $Y_1(\mathfrak{c}_i, \mathfrak{n})$ is a hyperbolic manifold admitting $\mathfrak{H}_F$ as a universal covering space with this group.

Put $\mathfrak{H}_F^* = \mathfrak{H}_F \coprod \mathbb{P}^1(F)$. The minimal compactification $Y_1(\mathfrak{c}_i, \mathfrak{n})^*$ of $Y_1(\mathfrak{c}_i, \mathfrak{n})$ is defined as $\Gamma_1(\mathfrak{c}_i, \mathfrak{n})\backslash\mathfrak{H}_F^*$. It is an analytic normal projective space whose boundary $\Gamma_1(\mathfrak{c}_i, \mathfrak{n})\backslash\mathbb{P}^1(F)$ is a finite union of points, called the *cusps*. We let $Y_1(\mathfrak{n})^* = \coprod_{i=1}^h Y_1(\mathfrak{c}_i, \mathfrak{n})^*$.

2.3. **Modular cycles.** Given an integral ideal $\mathfrak{f}$ and a fractional ideal $\mathfrak{c}$ of $F$, let $\Gamma$ be a congruence subgroup of $\mathrm{GL}_2(F)$ containing $s(E(\mathfrak{f}) \ltimes \mathfrak{c}^*)$.

**Lemma 2.1.** *Let $x \in F$ and let $\mathfrak{f}$ be the integral ideal of $F$ such that $x\mathfrak{o} + \mathfrak{c}^* = (\mathfrak{f}\mathfrak{c})^*$. The map*

$$F_\infty^+ \to \Gamma\backslash\mathfrak{H}_F, \quad y_\infty \mapsto \Gamma(y_\infty \underline{i} - x_\infty),$$

*factors through $E(\mathfrak{f})\backslash F_\infty^+$. The resulting map $C_x^\Gamma : E(\mathfrak{f})\backslash F_\infty^+ \to \Gamma\backslash\mathfrak{H}_F$ is finite and called the classical modular cycle.*

*Proof.* The map $C_x^\Gamma$ is well defined since for all $\epsilon \in E(\mathfrak{f})$, the element $\begin{pmatrix} \epsilon & (\epsilon-1)x \\ 0 & 1 \end{pmatrix} \in \Gamma$ sends $y_\infty \underline{i} - x_\infty$ to $\epsilon_\infty y_\infty \underline{i} - x_\infty$.

Let $\mathcal{C}$ be the closure of a Shintani cone in $-x_\infty + F_\infty^+ \underline{i}$ modulo $E(\mathfrak{f})$. To show that $C_x^\Gamma$ is finite, one has to show that for any given $y_\infty \in F_\infty^+$ the set $\Gamma \cdot (y_\infty \underline{i} - x_\infty) \cap \mathcal{C}$ is finite.

As well known $\Gamma \setminus \mathfrak{H}_F^*$ is separated for the Satake topology, hence each cusp has a neighbourhood which is disjoint from $\Gamma \cdot (y_\infty \underline{i} - x_\infty)$. Recall that a basis of neighbourhoods of the cusp at $\infty$ is given by sets of the form $\{z \in \mathfrak{H}_F | \prod_{\sigma\in\Sigma} \mathrm{Im}(z_\sigma) > A\}$ with $A > 0$. It follows that a basis of neighbourhoods of the cusp at $\infty$ (resp. at $-x$) in $\mathcal{C}$ is given by sets of the form $\{z \in \mathcal{C} | \prod_{\sigma\in\Sigma} \mathrm{Im}(z_\sigma) > A\}$ (resp. $\{z \in \mathcal{C} | \prod_{\sigma\in\Sigma} \mathrm{Im}(z_\sigma) < A'\}$) where $A, A' > 0$. It follows that there exist $A, A' > 0$ such that

$$\Gamma \cdot (y_\infty \underline{i} - x_\infty) \cap \mathcal{C} = \Gamma \cdot (y_\infty \underline{i} - x_\infty) \cap \{z \in \mathcal{C} | A' \leq \prod_{\sigma\in\Sigma} \mathrm{Im}(z_\sigma) \leq A\}.$$

Since $\{z \in \mathcal{C} | A' \leq \prod_{\sigma\in\Sigma} \mathrm{Im}(z_\sigma) \leq A\}$ is compact and since $\Gamma$ is acting properly discontinuously on $\mathfrak{H}_F^*$, it follows that $\Gamma \cdot (y_\infty \underline{i} - x_\infty) \cap \mathcal{C}$ is a finite set. $\square$

2.4. **Automorphic cycles.** We will now present the cycles introduced in [D1, §1] and establish some of their basic properties.

Let $\mathfrak{f}$ be an integral ideal of $F$. The the narrow ray class group $\mathcal{C}\ell_F^+(\mathfrak{f}) = F^\times \setminus \mathbb{A}^\times / U(\mathfrak{f}) F_\infty^+$ fits in the following short exact sequence:

$$1 \to E(\mathfrak{f}) \backslash F_\infty^+ \to \mathbb{A}^\times / F^\times U(\mathfrak{f}) \to \mathcal{C}\ell_F^+(\mathfrak{f}) \to 1. \tag{13}$$

Denote by $S$ be the set of places dividing $\mathfrak{f}$ and choose an idele $\varphi \in \mathbb{A}_F^\times$ generating $\mathfrak{f}$.

The map:

$$C_\varphi : \mathbb{A}^\times / F^\times U(\mathfrak{f}) \longrightarrow M(F) \setminus M(\mathbb{A}_F) / M(\mathfrak{o}) \ , \ y \mapsto M(F)(y, y\varphi_S^{-1}) M(\mathfrak{o}) \tag{14}$$

is well defined, since for all $\xi \in F^\times$ and $u \in U(\mathfrak{f})$ we have

$$(\xi y u, \xi y u \varphi_S^{-1}) = (\xi, 0)(y, y\varphi_S^{-1}) \left(u, (u-1)\varphi_S^{-1}\right),$$

where $(\xi, 0) \in M(F)$ whereas $(u, (u-1)\varphi_S^{-1}) \in M(\mathfrak{o})$.

**Definition 2.2.** For any $\eta \in \mathbb{A}^\times$ we define $C_{\varphi,\eta}$ as the composed map

$$E(\mathfrak{f}) \backslash F_\infty^+ \xrightarrow{\cdot\eta} \mathbb{A}^\times / F^\times U(\mathfrak{f}) \xrightarrow{C_\varphi} M(F) \setminus M(\mathbb{A}_F) / M(\mathfrak{o}).$$

**Lemma 2.3.** *If $\eta$ and $\eta'$ have the same image in $\mathcal{C}\ell_F^+(\mathfrak{f})$, then here is an orientation preserving homotopy between $C_{\varphi,\eta}$ and $C_{\varphi,\eta'}$.*

*For any $\varphi' \in \mathbb{A}_F^\times$ generating $\mathfrak{f}$, one has $C_{\varphi,\eta} = C_{\varphi',\eta\varphi'/\varphi}$.*

*Proof.* Suppose that $\eta' = \xi\eta u z_\infty$ with $\xi \in F^\times$, $u \in U(\mathfrak{f})$ and $z_\infty \in F_\infty^+$. For all $y_\infty \in F_\infty^+$

$$(\eta' y_\infty, \eta'\varphi_S^{-1}) = (\xi, 0)(\eta y_\infty z_\infty, \eta\varphi_S^{-1})(u, (u-1)\varphi_S^{-1}),$$

where $(u-1)\varphi_S^{-1} \in \widehat{\mathfrak{o}}$. Hence $C_{\varphi,\eta'}(E(\mathfrak{f})y_\infty) = C_{\varphi,\eta}(E(\mathfrak{f})y_\infty z_\infty)$ showing the first claim, since multiplication by $z_\infty \in F_\infty^+$ induces an orientation preserving homotopy equivalence of $E(\mathfrak{f})\backslash F_\infty^+$. The second claim follows from the identity

$$(\eta y_\infty, \eta\varphi_S^{-1}) = (\eta\varphi'\varphi^{-1} y_\infty, \eta\varphi'\varphi^{-1}{\varphi'_S}^{-1})(\varphi{\varphi'}^{-1}, 0),$$

since $\varphi{\varphi'}^{-1} \in U(\mathfrak{o})$, so that $(\varphi{\varphi'}^{-1}, 0) \in M(\mathfrak{o})$. □

**Definition 2.4.** For any $\eta \in \mathbb{A}^\times$ denote by $[\eta]$ its image in $\mathcal{C}\ell_F^+(\mathfrak{f})$. The automorphic cycle of level $\mathfrak{f}$ is defined as:

$$C_{\mathfrak{f}} = \sum_{\eta \in \mathcal{C}\ell_F^+(\mathfrak{f})} C_{\varphi,\eta}[\eta\varphi^{-1}].$$

Lemma 2.3 implies that, up to orientation preserving homotopy, $C_{\mathfrak{f}}$ only depends on $\mathfrak{f}$ and not on the particular choices of $\varphi$ or $\eta$.

For any open compact subgroup $K$ of $\mathrm{GL}_2(\mathbb{A}_F^{(\infty)})$ containing $s(M(\mathfrak{o}))$, $s$ induces a well defined map

$$s_K : M(F)\backslash M(\mathbb{A}_F)/M(\mathfrak{o}) \to Y_K.$$

**Definition 2.5.** The *automorphic cycle* $C_{\varphi,\eta}^K$ is defined as the composed map of $C_{\varphi,\eta}$ with the map $s_K$.

2.5. **Comparison of modular and automorphic cycles.** Let $K$ be an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_F^{(\infty)})$ containing $s(M(\mathfrak{o}))$. The connected components of $Y_K$ are in bijection with $\Gamma_i\backslash\mathfrak{H}_F$, $1 \le i \le h$ (see §2.2), where $\Gamma_i = \mathrm{GL}_2(F) \cap \left(\begin{smallmatrix}\eta_i & 0\\ 0 & 1\end{smallmatrix}\right) K \left(\begin{smallmatrix}\eta_i^{-1} & 0\\ 0 & 1\end{smallmatrix}\right) G_\infty^+$.

To be able to make the comparison, we define classical modular symbols taking values in the mirabolic group. Recall that $\eta_i$ generates the fractional ideal $\mathfrak{c}_i^*$.

**Lemma 2.6.** *For all $x \in F$ such that $x\mathfrak{o} + \mathfrak{c}_i^* = (\mathfrak{f}\mathfrak{c}_i)^*$, the map*

$$C(\eta_i, x) : E(\mathfrak{f})\backslash F_\infty^+ \to M(F)\backslash M(\mathbb{A}_F)/M(\mathfrak{o}), \quad y_\infty \mapsto (\eta_i y_\infty, -x_\infty)$$

*is well defined, injective and fits in the following commutative diagram:*

$$\begin{tikzcd} E(\mathfrak{f})\backslash F_\infty^+ \arrow[r, "C(\eta_i,x)"] \arrow[d, "C_x^{\Gamma_i}"'] & M(F)\backslash M(\mathbb{A}_F)/M(\mathfrak{o}) \arrow[d, "s_K"] \\ \Gamma_i\backslash\mathfrak{H}_F = \Gamma_i\backslash G_\infty^+/K_\infty^+ \arrow[r, "\left(\begin{smallmatrix}\eta_i & 0\\ 0 & 1\end{smallmatrix}\right)\cdot"] & Y_K \end{tikzcd} \tag{15}$$

*Proof.* Note that by definition $y_\infty$ (resp. $x_\infty$) is 1 (resp. 0) at all finite places.

For any $\epsilon \in E(\mathfrak{f})$ we have the following equalities in $M(\mathbb{A}_F)$:

$$(\epsilon, x(\epsilon-1)) \cdot (\eta_i y_\infty, -x_\infty) = (\epsilon \eta_i y_\infty, \epsilon^{(\infty)} x^{(\infty)} - x) =$$

$$= (\eta_i y_\infty \epsilon_\infty, -x_\infty)(\epsilon^{(\infty)}, \eta_i^{-1} x^{(\infty)}(\epsilon^{(\infty)} - 1)).$$

Since $(\epsilon, x(\epsilon-1)) \in M(F)$, while $\epsilon^{(\infty)} \in \widehat{\mathfrak{o}}^\times$, $\eta_i^{-1} x^{(\infty)} \in \varphi^{-1}\widehat{\mathfrak{o}}$ and $(\epsilon^{(\infty)} - 1) \in \varphi\widehat{\mathfrak{o}}$, one has

$$(\epsilon^{(\infty)}, \eta_i^{-1} x^{(\infty)}(\epsilon^{(\infty)} - 1)) \in M(\mathfrak{o})$$

which proves the first part of the lemma. The commutativity is straightforward.

For the injectivity one needs to show that if $(\eta_i y'_\infty, -x_\infty) \in (a,b)(\eta_i y_\infty, -x_\infty)M(\mathfrak{o})$ with $(a,b) \in M(F)$ then $y'_\infty y_\infty^{-1} \in E(\mathfrak{f})$. Projecting to $M(F_\infty)$ implies that $a_\infty = y'_\infty y_\infty^{-1} \in F_\infty^+$ and $b_\infty = (a_\infty - 1)x_\infty$, hence $b = (a-1)x$. Further projecting to $M(\mathbb{A}_F^{(\infty)})$ yields

$$(a^{(\infty)}, (a^{(\infty)} - 1)x^{(\infty)}\eta_i^{-1}) \in M(\mathfrak{o}),$$

hence $a^{(\infty)} \in U(\mathfrak{o}) \subset \widehat{\mathfrak{o}}$ and $a^{(\infty)} - 1 \in x^{-1}\eta_i\widehat{\mathfrak{o}}$. Since $\widehat{\mathfrak{o}} + x\eta_i^{-1}\widehat{\mathfrak{o}} = \varphi^{-1}\widehat{\mathfrak{o}}$ this implies that

$$a^{(\infty)} - 1 \in \widehat{\mathfrak{o}} \cap x^{-1}\eta_i\widehat{\mathfrak{o}} = \varphi\widehat{\mathfrak{o}}$$

showing that $a \in F^\times \cap (U(\mathfrak{o}) \cap (1 + \varphi\widehat{\mathfrak{o}}))F_\infty^+ = F^\times \cap U(\mathfrak{f})F_\infty^+ = E(\mathfrak{f})$ as desired. $\square$

**Proposition 2.7.** *Given $\eta \in \mathbb{A}_F^{(\infty)\times}$ there exists a unique $1 \le i \le h$ such that $\eta$ and $\eta_i$ map to the same element of $\mathcal{C}\ell_F^+$, i.e. $\eta = a^{(\infty)}\eta_i u$ with $a \in F_+^\times$ and $u \in U(\mathfrak{o})$. For $S$ and $\varphi$ as in §2.4 and for any $x \in (\mathfrak{f}\mathfrak{c}_i)^*$ whose image in $(\mathfrak{f}\mathfrak{c}_i)^*/\mathfrak{c}_i^*$ equals $u\eta_i\varphi_S^{-1}$, the multiplication by $a_\infty \in F_\infty^+$ induces an orientation preserving homotopy between $C_{\varphi,\eta}$ (resp. $C_{\varphi,\eta}^K$) and $C(\eta_i, x)$ (resp. $\left(\begin{smallmatrix}\eta_i & 0\\ 0 & 1\end{smallmatrix}\right) \cdot C_x^{\Gamma_i}$). In other terms, there is a commutative diagram:*

(16)
$$\begin{tikzcd}
E(\mathfrak{f})\backslash F_\infty^+ \arrow[rr, "\cdot\eta"] \arrow[drr, "C_{\varphi,\eta}"] & & E(\mathfrak{f})\backslash F_\infty^+\eta \arrow[d, "C_\varphi"] \\
E(\mathfrak{f})\backslash F_\infty^+ \arrow[u, "\cdot a_\infty"] \arrow[rr, "C(\eta_i,x)"'] & & M(F)\backslash M(\mathbb{A}_F)/M(\mathfrak{o})
\end{tikzcd}.$$

*Proof.* Since $\eta = a^{(\infty)}\eta_i u$, a direct computation shows the following identity in $M(\mathbb{A}_F)$:

$$(\eta y_\infty a_\infty, \eta\varphi_S^{-1}) = (a, ax)(\eta_i y_\infty, -x_\infty)(u, u\varphi_S^{-1} - \eta_i^{-1}x^{(\infty)})$$

where $(a, ax) \in M(F)$. Moreover the assumption on $x$ implies that $u\varphi_S^{-1} - \eta_i^{-1}x^{(\infty)} \in \widehat{\mathfrak{o}}$, so that $(u, u\varphi_S^{-1} - \eta_i^{-1}x^{(\infty)}) \in M(\mathfrak{o})$. This proves the commutativity of the lower triangle in the diagram, while the commutativity of the other triangle follows directly from the definition of $C_{\varphi,\eta}$. Finally, the comparison between $C_{\varphi,\eta}^K$ and $C_x^{\Gamma_i}$ follows from (15), (16) and Definition 2.5. $\square$

**Corollary 2.8.** *Up to orientation preserving homotopy the cycle $C(\eta_i, x)$ depends only on the image of $x$ in the group:*

$$((\mathfrak{f}\mathfrak{c}_i)^*/\mathfrak{c}_i^*)^\times /E(\mathfrak{o}).$$

**Remark 2.9.** Note that whereas the automorphic cycles of level $\mathfrak{f}$ are indexed by the middle term of the short exact sequence:

$$1 \to (\mathfrak{o}/\mathfrak{f})^\times/E(\mathfrak{o}) \to \mathcal{C}\ell_F^+(\mathfrak{f}) \to \mathcal{C}\ell_F^+ \to 1, \tag{17}$$

the modular ones are indexed by elements of $\mathcal{C}\ell_F^+ \times (\mathfrak{o}/\mathfrak{f})^\times/E(\mathfrak{o})$. In fact the elements of $\mathcal{C}\ell_F^+$ are represented by $\eta_i$, $1 \le i \le h$, while multiplication by $\eta_i^{-1}\varphi$ induces an isomorphism

$$((\mathfrak{f}\mathfrak{c}_i)^*/\mathfrak{c}_i^*)^\times/E(\mathfrak{o}) \xrightarrow{\sim} (\mathfrak{o}/\mathfrak{f})^\times/E(\mathfrak{o}).$$

Therefore the automorphic cycles are more intrinsic than the modular cycles.

In view of Lemma 2.1, Proposition 2.7 has another consequence.

**Corollary 2.10.** *The automorphic cycle $C^K_{\varphi,\eta}$ (see Definition 2.5) is finite as a map.*

## 3. $p$-adic $L$-functions for Hilbert modular forms

3.1. **Cohomological weights.** The characters of the $\mathbb{Q}$-torus $F^\times$ can be identified with $\mathbb{Z}[\Sigma]$ as follows: for any $k = \sum_{\sigma\in\Sigma} k_\sigma\sigma \in \mathbb{Z}[\Sigma]$ and for any $\mathbb{Q}$-algebra $A$ splitting $F^\times$, we consider the character $k \in (F\otimes_{\mathbb{Q}} A)^\times \mapsto x^k = \prod_{\sigma\in\Sigma}\sigma(x)^{k_\sigma} \in A^\times$. The norm character $\mathrm{N}_{F/\mathbb{Q}} : F^\times \to \mathbb{Q}^\times$ then corresponds to the element $t = \sum_{\sigma\in\Sigma}\sigma \in \mathbb{Z}[\Sigma]$.

Any algebraic character of the diagonal torus of $\mathrm{GL}_2(F)$ is of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right) \mapsto a^k d^{k'}$ for some $(k,k') \in \mathbb{Z}[\Sigma]^2$. Characters such that $k_\sigma \ge k'_\sigma$ for all $\sigma\in\Sigma$ are called dominant with respect to upper triangular Borel and parametrise the irreducible representation of the algebraic $\mathbb{Q}$-group $\mathrm{GL}_2(F)$. Explicitly, for any $\mathbb{Q}$-algebra $A$ splitting $F^\times$, the irreducible representation of $\mathrm{GL}_2(A)$ of highest weight $(k,k')$ is given by

$$\bigotimes_{\sigma\in\Sigma}\left(\mathrm{Sym}_\sigma^{k_\sigma - k'_\sigma}\otimes \mathrm{Det}_\sigma^{k'_\sigma}\right)(A^2).$$

**Definition 3.1.** A dominant weight of $T$ is *cohomological* if it is of the form $(\frac{\mathsf{w}t+k}{2}, \frac{\mathsf{w}t-k}{2})$ where $(k,\mathsf{w}) \in \mathbb{Z}[\Sigma]\times\mathbb{Z}$ is such that for all $\sigma\in\Sigma$ we have $k_\sigma \ge 0$ and $k_\sigma \equiv \mathsf{w} \pmod 2$. We denote

$$V_{k,\mathsf{w}} = \bigotimes_{\sigma\in\Sigma}\mathrm{Sym}_\sigma^{k_\sigma}\otimes\mathrm{Det}_\sigma^{(\mathsf{w}t-k_\sigma)/2}$$

the corresponding irreducible representation of $G$. For any $\mathbb{Q}$-algebra $A$ splitting $F^\times$ write $V_{k,\mathsf{w}}(A)$ for its $A$-valued points.

Note that a dominant weight is cohomological if, and only if, the central character of the corresponding representations of $\mathrm{GL}_2(F)$ factors through the norm. Under this assumption the center of any (sufficiently small) congruence subgroup of $\mathrm{GL}_2(F)$ will act trivially, ensuring the existence of a local system $\mathcal{V}_{k,\mathsf{w}}$ on $Y_K$ attached to $V_{k,\mathsf{w}}$.

The left $A[\mathrm{GL}_2(F \otimes_{\mathbb{Q}} A)]$-module $V_{k,\mathsf{w}}(A)$ can be realised as the space of polynomials of degree $(k_\sigma)_{\sigma\in\Sigma}$ in $z = (z_\sigma)_{\sigma\in\Sigma}$ over $A$ on which $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(F \otimes_{\mathbb{Q}} A) \simeq \mathrm{GL}_2(A)^\Sigma$ acts by:

$$(\gamma \cdot P)(z) = (ad-bc)^{(\mathsf{w}t-k)/2}(a-cz)^k P\left(\frac{dz-b}{a-cz}\right). \tag{18}$$

3.2. **Local systems and cohomology.** Consider a left $\mathrm{GL}_2(F)$-module $V$ such that

$$F^\times \cap KF_\infty^\times \text{ acts trivially on } V. \tag{19}$$

For $K$ sufficiently small we have $\mathrm{GL}_2(F) \cap gKK_\infty^+ g^{-1} = F^\times \cap KF_\infty^\times$ which by (19) acts trivially on $V$. Therefore one has a local system

$$\mathrm{GL}_2(F)\backslash(\mathrm{GL}_2(\mathbb{A}_F) \times V)/KK_\infty^+ \to Y_K$$

with left $\mathrm{GL}_2(F)$-action and right $KK_\infty^+$-action given by:

$$\gamma(g,v)k = (\gamma gk, \gamma\cdot v).$$

We will denote by $\mathcal{V}$ the corresponding sheaf of locally constant sections on $Y_K$ and will consider the usual (resp. compactly supported) cohomology groups $\mathrm{H}^i(Y_K,\mathcal{V})$ (resp. $\mathrm{H}^i_c(Y_K,\mathcal{V})$). In particular, for any cohomological weight $(k,\mathsf{w})$ and any $\mathbb{Q}$-algebra $A$ splitting $F^\times$ we will denote $\mathcal{V}_{k,\mathsf{w}}(A)$ the sheaf associated to $V_{k,\mathsf{w}}(A)$.

There is another construction of sheaves. Namely, given a left $K$-module $V$ satisfying (19), one can consider the local system

$$\mathrm{GL}_2(F)\backslash(\mathrm{GL}_2(\mathbb{A}_F) \times V)/K_\infty^+ K \to Y_K$$

with left $\mathrm{GL}_2(F)$-action and right $KK_\infty^+$-action given by:

$$\gamma(g,v)k = (\gamma gk, k^{-1}\cdot v).$$

When the actions of $\mathrm{GL}_2(F)$ and $KK_\infty^+$ on $V$ in the above two definitions can be extended compatibly into a left action of $\mathrm{GL}_2(\mathbb{A}_F)$, then the resulting two local systems are isomorphic by $(g,v)\mapsto(g,g^{-1}\cdot v)$.

We will be particularly interested in the case where $A$ is a $p$-adic field and both $\mathrm{GL}_2(F)$ and $K_p$ act compatibly on $V_{k,\mathsf{w}}(A)$. The $\mathrm{GL}_2(F)$-action will be used to define $\mathrm{H}^i(Y_K,\mathcal{V}_{k,\mathsf{w}}^\vee(\mathbb{C}))$ which admits an interpretation in terms of automorphic forms on $\mathrm{GL}_2(\mathbb{A}_F)$ while the $K_p$-action will be used to interpolate $\mathrm{H}^i(Y_K,\mathcal{V}_{k,\mathsf{w}}(L))$ where $L$ is a $p$-adic field.

3.3. **Overconvergent cohomology of Hilbert modular varieties.** Given a cohomological weight $(k, \mathsf{w})$ and a $p$-adic field $L$ containing the Galois closure of $F$, we let $D_{k,\mathsf{w}}$ denote the space $D(\mathfrak{o}\otimes\mathbb{Z}_p, L)$ of $L$-values distributions on $\mathfrak{o}\otimes\mathbb{Z}_p$ (see §1.7) on which the Iwahori subgroup $I \subset \mathrm{GL}_2(\mathfrak{o}\otimes\mathbb{Z}_p)$ acts on the right as follows:

$$\mu_{|\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)}(f(z)) = \mu\left((ad-bc)^{(\mathsf{w}t-k)/2}(a-cz)^k f\left(\frac{dz-b}{a-cz}\right)\right). \tag{20}$$

Furthermore, for $\mathfrak{p} \mid p$ we fix an uniformizer $\varpi_\mathfrak{p}$ of $F_\mathfrak{p}$ and define:

$$\mu_{|\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_\mathfrak{p}\end{smallmatrix}\right)}(f(z)) = \mu(f(\varpi_\mathfrak{p} z)), \tag{21}$$

where $\varpi_\mathfrak{p}\in\mathfrak{o}\otimes\mathbb{Z}_p$ is considered to be 1 at all components $\mathfrak{p}' \mid p$, $\mathfrak{p}'\neq\mathfrak{p}$.

The actions (20) and (21) extend compatibly into an action on $D_{k,\mathsf{w}}$ of the monoid $\Delta$ generated by $I$ and the matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_\mathfrak{p}\end{smallmatrix}\right)$, for $\mathfrak{p}\mid p$.

For any open compact subgroup $K$ of $\mathrm{GL}_2(\mathbb{A}_F^{(\infty)})$ whose image into $\mathrm{GL}_2(F\otimes_\mathbb{Q}\mathbb{Q}_p)$ is contained in $I$ one can associate to $D_{k,\mathsf{w}}$ a local system $\mathcal{D}_{k,\mathsf{w}}$ on $Y_K$ and consider the compactly supported sheaf cohomology groups $\mathrm{H}^i_c(Y_K, \mathcal{D}_{k,\mathsf{w}})$.

As in §1.10 the element $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p\end{smallmatrix}\right)\in\Delta$ induces a compact operator $U_p$ on $D_{k,\mathsf{w}}$ and $\mathrm{H}^i_c(Y_K, \mathcal{D}_{k,\mathsf{w}})$ admits a slope decomposition with respect to it. As for $\mathrm{H}^i_c(Y_K, \mathcal{V}_{k,\mathsf{w}}(L))$ we consider slope decomposition with respect to the operator $U_p^0 = p^{(k-\mathsf{w}t)/2}U_p$.

The natural restriction map:

$$D_{k,\mathsf{w}} \to V_{k,\mathsf{w}}(L), \quad \mu\mapsto P(\mu)(z) = \int_{\mathfrak{o}\otimes\mathbb{Z}_p}(z-x)^k d\mu(x) \tag{22}$$

is $I$-equivariant. Moreover the induced homomorphism:

$$\mathrm{H}^i_c(Y_K, \mathcal{D}_{k,\mathsf{w}}) \to \mathrm{H}^i_c(Y_K, \mathcal{V}_{k,\mathsf{w}}(L)) \tag{23}$$

is compatible with slope decompositions with respect to $U_p$ for $\mathrm{H}^i_c(Y_K, \mathcal{D}_{k,\mathsf{w}})$ and with respect to $U_p^0$ for $\mathrm{H}^i_c(Y_K, \mathcal{V}_{k,\mathsf{w}}(L))$. Stevens' Theorem 1.13 has the following generalisation when $\mathbb{Q}$ is replaced by an arbitrary totally real number field $F$.

**Theorem 3.2** (Barrera [B]). *For any $h\in\mathbb{Q}_+$ such that $h < k_\sigma + 1$ for all $\sigma\in\Sigma$, (23) induces an isomorphism:*

$$\mathrm{H}^i_c(Y_K, \mathcal{D}_{k,\mathsf{w}})^{\leq h} \xrightarrow{\sim} \mathrm{H}^i_c(Y_K, \mathcal{V}_{k,\mathsf{w}}(L))^{\leq h}.$$

3.4. **$p$-adic $L$-functions for Hilbert modular forms.** In this final section we give a brief sketch of Barrera's construction of $p$-adic $L$-functions for Hilbert modular forms based on the cycles considered in §2.

Consider a cuspidal cohomological automorphic representation $\pi$ of $\mathrm{GL}_2(\mathbb{A}_F)$ of conductor $\mathfrak{n}$ and of infinity type $(k+2t, \mathsf{w})$, where $\mathsf{w}$ denotes the purity weight of $\pi$.

According to Deligne [De], the integer 1 is critical for the motive attached to $\pi$ exactly when $(k, \mathsf{w})$ is critical in the sense of the following definition.

**Definition 3.3.** A cohomological weight $(k, \mathsf{w})$ is *critical* if $|\mathsf{w}| \leq \min_{\sigma\in\Sigma}(k_\sigma)$.

Let $f$ be a $p$-stabilisation of the new-vector in $\pi$, so that $U_{\mathfrak{p}}f = \alpha_{\mathfrak{p}}f$ for all primes $\mathfrak{p}$ dividing $p$. Let $K$ be the subgroup of $K_1(\mathfrak{n})$ obtained by intersecting its $p$-component with $I$. Using a result of Matsushima-Shimura and Harder, as worked out in Hida [H], there exists a class $\delta_f^+$ in the complex line $\mathrm{H}^d_{\mathrm{cusp}}(Y_K, \mathcal{V}^\vee_{k,\mathsf{w}}(\mathbb{C}))[f]^+$. Assume further that $L$ contains all the Hecke eigenvalues of $f$. Dividing $\delta_f^+$ by a period $\Omega_f^+ \in \mathbb{C}^\times$ yields a class

$$\phi_f \in \mathrm{H}^d_c(Y_K, \mathcal{V}_{k,\mathsf{w}}(L))[f]^+.$$

Assume the following non-critical condition:

$$h = \sum_{\sigma\in\Sigma} \frac{k_\sigma - \mathsf{w}}{2} + \sum_{\mathfrak{p}|p} v_p(\iota_p(\alpha_{\mathfrak{p}}))e_{\mathfrak{p}} < \min_{\sigma\in\Sigma}(k_\sigma + 1), \tag{24}$$

where $e_{\mathfrak{p}}$ denotes the ramification index, so that $(p) = \prod_{\mathfrak{p}|p} \mathfrak{p}^{e_{\mathfrak{p}}}$.

By Theorem 3.2 there exists a unique class

$$\Phi_f \in \mathrm{H}^d_c(Y_K, \mathcal{D}_{k,\mathsf{w}})[f]^+$$

which maps to $\phi_f$ under (23).

Evaluating $\Phi_f$ on the modular cycles on $Y_K$ of $p$-power conductor (see Definition 2.4), Barrera constructs a distribution $\mu_f \in D(\mathcal{C}\ell_F^+(p^\infty), L)$ and proves that it is $h$-admissible. Using the computations performed in [D1, §2] he proves the following theorem.

**Theorem 3.4** (Barrera [B])**.** *For any finite order Hecke character $\chi : \mathcal{C}\ell_F^+(p^\infty) \to L^\times$ such that $\chi_\sigma(-1) = 1$ for each $\sigma \in \Sigma$ we have:*

$$\mu_f(\chi) = \iota_p\left(\frac{L^{(p)}(\pi\otimes\chi, 1)\tau(\chi)}{\Omega_f^+}\right)\prod_{\mathfrak{p}|p} Z_{\mathfrak{p}},$$

*where $L^{(p)}(\pi\otimes\chi, s)$ is the L-function of $\pi$ twisted by $\chi$ without the Euler factor at all places dividing $p$, $\tau(\chi)$ is the Gauss sum, $d_{\mathfrak{p}}$ is the valuation of the different of $F_{\mathfrak{p}}/\mathbb{Q}_p$, and:*

$$Z_{\mathfrak{p}} = \begin{cases} \iota_p(\alpha_{\mathfrak{p}})^{-\mathrm{cond}(\chi_{\mathfrak{p}})} & \text{, if } \chi_{\mathfrak{p}} \text{ is ramified, and} \\ \frac{1-\iota_p(\alpha_{\mathfrak{p}})^{-1}\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})^{-1}\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})^{-1}}{1-\iota_p(\alpha_{\mathfrak{p}})\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})}\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})^{-d_{\mathfrak{p}}} & \text{, otherwise.} \end{cases}$$

Note that (in the non-ordinary case) the interpolation property proved in Theorem 3.4 does not guarantee the uniqueness of $\mu_f$. This problem is settled in [BDJ].

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Univ. Lille, CNRS, UMR 8524 – Laboratoire Paul Painlevé, 59000 Lille, France

*E-mail address*: `mladen.dimitrov@math.univ-lille1.fr`