ARITHMETIC ASPECTS OF HILBERT MODULAR FORMS AND VARIETIES

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Abstract. Hilbert modular forms and varieties are the natural generalization of elliptic modular forms and curves, when the ground field of rational numbers is replaced by a totally real number field. The aim of these notes is to present the basics of their arithmetic theory and to describe some of the recent results in the area. A special emphasis will be put on the following two subjects: images of Galois representations associated to Hilbert modular forms and cohomology of Hilbert modular varieties with integral coefficients.

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Contents

1. Hilbert modular forms

Let F be a totally real number field of degree $d > 1$, ring of integers $\mathcal{O}_F$ and denote by J_F the set of all embeddings of F into $\mathbb{R}$.

The torus $F^\times$ is quasi-split over $\mathbb{Q}$ and its group of characters can be identified with $\mathbb{Z}[J_F]$ as follows: for any $k = \sum_{\tau \in J_F} k_\tau \tau \in \mathbb{Z}[J_F]$ and for any $\mathbb{Q}$-algebra A splitting $F^\times$, we consider the character $x \in (F \otimes_{\mathbb{Q}} A)^\times \mapsto x^k := \prod_{\tau \in J_F} \tau(x)^{k_\tau} \in A^\times$. The norm character $\mathrm{N_{F/\mathbb{Q}}} : F^\times \to \mathbb{Q}^\times$ then corresponds to the element $t := \sum_{\tau \in J_F} \tau \in \mathbb{Z}[J_F]$.

1.1. **Congruence subgroups.** Denote by $\widehat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$. The ring $\mathbb{A}$ of adeles of F is the product of finite adeles $\mathbb{A}_f = F \otimes \widehat{\mathbb{Z}}$ with infinite adeles $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. Denote by F_∞^+ the open cone of totally positive elements in $F_\infty^\times$.

For an open compact subgroup U of $(\mathcal{O}_F \otimes \widehat{\mathbb{Z}})^\times$ we denote by $\mathcal{C}_U$ (resp. $\mathcal{C}_U^+$) the class group $\mathbb{A}^\times/F^\times U F_\infty^\times$ (resp. the narrow class group $\mathbb{A}^\times/F^\times U F_\infty^+$).

For an ideal integral ideal $\mathcal{N}$ of $\mathcal{O}_F$, we consider the following open compact subgroup of $\mathbb{A}_f^\times$:

$$U(\mathcal{N}) = \left\{ x \in (\mathcal{O}_F \otimes \widehat{\mathbb{Z}})^\times \,\middle|\, x - 1 \in \mathcal{N} \otimes \widehat{\mathbb{Z}} \right\},$$

and following open compact subgroups of $\mathrm{GL}_2(\mathbb{A}_f)$:

$$K_0(\mathcal{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F \otimes \widehat{\mathbb{Z}}) \,\middle|\, c \in \mathcal{N} \otimes \widehat{\mathbb{Z}} \right\},$$

$$K_1(\mathcal{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathcal{N}) \,\middle|\, d \in U(\mathcal{N}) \right\},$$

$$K_{11}(\mathcal{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_1(\mathcal{N}) \,\middle|\, a \in U(\mathcal{N}) \right\} \text{ and}$$

$$K(\mathcal{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{11}(\mathcal{N}) \,\middle|\, b \in \mathcal{N} \otimes \widehat{\mathbb{Z}} \right\}.$$

1.2. **Hilbert modular forms as automorphic forms on $\mathrm{GL}_2(\mathbb{A})$.** The group $G_\infty^+ := F_\infty^\times \mathrm{SL}_2(F_\infty)$ acts by linear fractional transformations on the d-fold product of upper half planes:

$$\mathfrak{H}_F := F_\infty + (1 \otimes \sqrt{-1}) F_\infty^+ \subset F \otimes_{\mathbb{Q}} \mathbb{C}.$$

Denote by K_∞^+ the stabilizer of $1 \otimes \sqrt{-1}$ in G_∞^+.

Definition 1.1. A weight $(k, w_0) \in \mathbb{Z}[J_F] \times \mathbb{Z}$ is *arithmetic* (or cohomological) if for all $\tau \in J_F$, $k_\tau \geq 2$ and $k_\tau \equiv w_0 \pmod 2$.

Definition 1.2. The space $M_{k,w_0}(\Gamma)$ of classical Hilbert modular forms of arithmetic weight (k, w_0) and level Γ (a congruence subgroup of $\mathrm{GL}_2(F) \cap \mathrm{GL}_2(\mathbb{A}_f) G_\infty^+$) consists of all holomorphic functions $f : \mathfrak{H}_F \to \mathbb{C}$ such that for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$f\left(\frac{az+b}{cz+d}\right) = \det(\gamma)^{(w_0 t - k)/2} j(\gamma, z)^k f(z),$$

where $j(\gamma, z) = cz + d \in (F \otimes \mathbb{C})^\times$ is the usual automorphic cocycle.
The subspace $S_{k,w_0}(\Gamma)$ of cuspforms consists of those f vanishing at all cusps of $\mathfrak{H}_F$.

Note that the action of Γ on $\mathfrak{H}_F$ is via all the embedding of F in $\mathbb{R}$, hence cannot be decomposed as a product.

The spaces $M_{k,w_0}(\Gamma)$ and $S_{k,w_0}(\Gamma)$ are finite dimensional $\mathbb{C}$-vector spaces, but as it will become clear, they are not stable under the action of Hecke operators in general, which motivates the use of the following adelic definition.

Definition 1.3. The space $M_{k,w_0}(K)$ of (adelic) Hilbert modular forms of weight (k, w_0) and level K (an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$) consists of all functions $f : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ which are left $\mathrm{GL}_2(F)$-invariant, right K-invariant and such that for all $g \in \mathrm{GL}_2(\mathbb{A}_f)$ the function

$$\gamma \in G_\infty^+ \mapsto \det(\gamma)^{(w_0 t-k)/2} j(\gamma, 1 \otimes \sqrt{-1})^k f(g\gamma),$$

factors through a homolorphic function on $G_\infty^+/K_\infty^+ \simeq \mathfrak{H}_F$, denoted by f_g.

If moreover $\int_{F\backslash \mathbb{A}} f\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\right) dx = 0$ for all $g \in \mathrm{GL}_2(\mathbb{A})$ then f is called a cuspform, and we denote by $S_{k,w_0}(K)$ the space of cuspforms.

We say that a form f has central character ψ, a Hecke characters of F of infinity type $-w_0 t$, if $f(y\cdot) = \psi(y)f$ for all $y \in \mathbb{A}^\times$, and we denote by $S_{k,w_0}(K, \psi) \subset S_{k,w_0}(K)$ the corresponding subspace.

Using the strong approximation theorem for GL_2, one can compare the adelic and the classical definition as follows. Choose elements $g_i \in \mathrm{GL}_2(\mathbb{A}_f)$, $1 \le i \le h$ such that $(\det(g_i))_{1\le i\le h}$ forms a set of representatives of $\mathcal{C}^+_{\det(K)}$. Then the map $f \mapsto (f_{g_i})_{1\le i\le h}$ induces isomorphisms

$$M_{k,w_0}(K) \simeq \bigoplus_{1\le i\le h} M_{k,w_0}(\Gamma_{g_i}) \text{ and } S_{k,w_0}(K) \simeq \bigoplus_{1\le i\le h} S_{k,w_0}(\Gamma_{g_i}),$$

where for $g \in \mathrm{GL}_2(\mathbb{A}_f)$ we put $\Gamma_g := \mathrm{GL}_2(F) \cap gKg^{-1}G_\infty^+$.

1.3. **Hecke operators and newforms.** The space $M_{k,w_0}(K)$ admits left action of the Hecke algebra $\mathcal{C}_c(K\backslash \mathrm{GL}_2(\mathbb{A}_f)/K)$ of bi-K-invariant compactly supported functions on $\mathrm{GL}_2(\mathbb{A}_f)$. In more concrete terms, for every $g \in \mathrm{GL}_2(\mathbb{A}_f)$, the Hecke operator $[KgK]$, corresponding to the characteristic function of KgK, sends f to $\sum_i f(\cdot g_i)$, where $KgK = \coprod_i g_i K$. The subspace $S_{k,w_0}(K)$ is stable under this action.

For a prime v, let ϖ_v denote an uniformizer of F_v. The standard Hecke operator $[K_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K_v]$ is denoted by T_v, if K_v is a maximal open compact subgroup of $\mathrm{GL}_2(F_v)$, and by U_v, otherwise.

The Hecke algebra is *not* commutative in general and one rather works with the commutative subalgebra generated by the standard Hecke operators and the center.

There is a direct sum decomposition:

$$S_{k,w_0}(K) \simeq \bigoplus_{\psi} S_{k,w_0}(K, \psi),$$

where ψ runs over all Hecke characters of level $K \cap \mathbb{A}_f^\times$ and type $-w_0 t$ at infinity.

The Peterson inner product

$$(f_1, f_2) := \int_{\mathrm{GL}_2(F)\,\mathbb{A}^\times\backslash \mathrm{GL}_2(\mathbb{A})} f_1(g)\overline{f_2(g)} |\det(g)|_{\mathbb{A}}^{w_0} dg$$

endows $S_{k,w_0}(K, \psi)$ with a structure of a hermitian space with respect to which the operators T_v are normal. It follows that $S_{k,w_0}(K, \psi)$ can be decomposed as a direct sum of eigenspaces for all the T_v's. Note that while the

U_v's form a commutative family of operators preserving this decomposition, they are not semi-simple in general, hence $S_{k,w_0}(K,\psi)$ does not always have a basis of eigenforms for all standard Hecke operators.

The theory of Atkin and Lehner addresses this problem for $K = K_1(\mathcal{N})$. More precisely, if one considers the subspace of *primitive* forms in $S_{k,w_0}(K_1(\mathcal{N}),\psi)$ (those orthogonal with respect to the Peterson inner product to all forms coming from lower level), their theory implies that the standard Hecke operators preserve this space and are semi-simple. The Strong Multiplicity One Theorem states that a primitive form f which is an eigenform for T_v, $v \nmid \mathcal{N}$, is uniquely determined, up to a multiple, by its eigenvalues $c(f,v)$ (hence it is necessarily an eigenform for U_v, $v \mid \mathcal{N}$, too).

Recall that the Weak Multiplicity One Theorem for GL_2 states that an element of $S_{k,w_0}(K_1(\mathcal{N}),\psi)$ which is an eigenform for T_v ($v \nmid \mathcal{N}$) and for U_v ($v \mid \mathcal{N}$) is uniquely determined, up to a multiple, by its eigenvalues.

A suitably normalized primitive eigenform in $S_{k,w_0}(K_1(\mathcal{N}),\psi)$ is called a *newform.*

There is a natural bijection between newforms f in $S_{k,w_0}(K_1(\mathcal{N}),\psi)$ and cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$, of conductor $\mathcal{N}$, central character ψ and such that π_∞ belongs to the holomorphic discrete series of arithmetic weight (k,w_0) (see [3]). It is uniquely characterized by the property that for all $v \nmid \mathcal{N}$, $c(f,v)$ is the eigenvalue of T_v acting on the new line $\pi_v^{\mathrm{GL}_2(\mathcal{O}_{F,v})}$.

2. Galois representations associated to Hilbert modular forms

The absolute Galois group of a field L is denoted by G_L.

Recall that we have an exact sequence $1 \to I_v \to G_{F_v} \to \widehat{\mathbb{Z}} \to 1$ and that the Weil group W_{F_v} is defined as the inverse image of $\mathbb{Z}$. The local class field theory gives an isomorphism between $F_v^\times$ and the maximal abelian quotient of W_{F_v}, that we normalize so that ϖ_v is sent to a geometric Frobenius Frob_v.

2.1. **Galois representations.** Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ of conductor $\mathcal{N}$, such that π_∞ belongs to the holomorphic discrete series of arithmetic weight (k,w_0). The central character ψ of π is a Hecke character of weight $-w_0 t$, that is $\psi|\cdot|_{\mathbb{A}}^{w_0}$ is of finite order. In classical terms, π corresponds to a Hilbert modular newform f over F of level $\mathcal{N}$, weight (k,w_0) and central character ψ (see §1.3).

For a prime p and an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ one can associate to π and ι_p a p-adic representation (*cf* [25, 26]):

$$\rho_{\pi,p} : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p), \tag{1}$$

which is irreducible, totally odd, unramified outside $\mathcal{N}p$ and characterized by the property that for each prime v not dividing $\mathcal{N}p$ we have $\mathrm{tr}(\rho_{\pi,p}(\mathrm{Frob}_v)) = \iota_p(c(f,v))$. Moreover $\det \rho_{\pi,p} = \psi\chi_p$, where χ_p denotes the p-adic cyclotomic character.

At all places v not dividing p, $\rho_{\pi,p}|_{W_{F_v}}$ is related to π_v by the local Langlands correspondence (see [5]).

The embedding ι_p defines a partition $J_F = \coprod_v J_{F_v}$, where v runs over the primes of F dividing p and J_{F_v} denotes the set of embeddings of F_v in $\overline{\mathbb{Q}}_p$.

At places v dividing p, the representation $\rho_{\pi,p}|_{G_{F_v}}$ is known to be de Rham of Hodge-Tate weights $(\frac{w_0-k_\tau}{2}+1, \frac{w_0+k_\tau}{2})_{\tau\in J_{F_v}}$ and crystalline for p large enough (*cf* [1], [2], [14] and [19]).

2.2. **Images of Galois representations.**

The representation $\rho_{\pi,p}$ is defined over the ring of integers $\mathcal{O}$ of a finite extension E of $\mathbb{Q}_p$. Let $\overline{\rho}_{\pi,p} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be the semi-simplification of the reduction of $\rho_{\pi,p}$ modulo a uniformizer ϖ of $\mathcal{O}$.

The following theorem established in [7, §3] generalizes results of Serre and Ribet [22] on classical modular forms to the case of Hilbert modular forms.

Theorem 2.1. (i) *For all but finitely many primes p, $\overline{\rho}_{\pi,p}$ is irreducible.*

(ii) *Assume that π has no CM. Then for all but finitely many primes p, the image of $\overline{\rho}_{\pi,p}$ contains (a conjugate of) $\mathrm{SL}_2(\mathbb{F}_p)$.*

(iii) *Assume that π has no CM and that it is not a twist of a base change. Then for all but finitely many primes p, $\bar{\rho} = \overline{\rho}_{\pi,p}$ fulfills the following condition:*

($\mathbf{LI}_{\bar\rho}$) *the image of $\bar\rho$ contains $\mathrm{SL}_2(\mathbb{F}_p)$ and none of its twists extends to a representation of $G_{F'}$ for any strict subfield F' of F.*

For the determination of image of $\rho_{\pi,p}$ itself, we refer to [20, Appendix B] where the author adapts the classical theory of inner twists for Hilbert modular forms.

2.3. **Modularity lifting theorems.**

A conjecture of Fontaine and Mazur states (see [11] for $F=\mathbb{Q}$) that any two-dimensional, irreducible, totally odd p-adic representation of G_F unramified outside a finite set of primes and de Rham at all primes v dividing p, with distinct Hodge-Tate weights for each $F_v \hookrightarrow \overline{\mathbb{Q}}_p$, is *automorphic*, that is to say can be obtained as in §2.1.

In the approach initiated by Wiles [30] and Taylor-Wiles [28], and extended by Diamond [6] and Fujiwara [12], this conjecture splits naturally in two parts. The first is a conjecture of Buzzard, Diamond and Jarvis [4], generalizing Serre's modularity conjecture (now a theorem of Khare and Wintenberger [16, 17]) to totally real number fields, stating that every two-dimensional, irreducible, totally odd representation $\bar\rho$ of G_F over a finite field has an automorphic lift. Since there is no a general result in this direction, we consider the following assumption:

($\mathbf{Mod}_{\bar\rho}$) p is unramified in F and there exists a cuspidal automorphic representation π of level prime to p and weight (k, w_0) such that $w_0 = \max\limits_{\tau\in J_F}(k_\tau - 2)$, $p-1 > \sum_{\tau\in J_F} \frac{w_0+k_\tau}{2}$ and $\overline{\rho}_{\pi,p} \simeq \bar\rho$.

The second part of the conjecture states that if $\bar{\rho}$ has an automorphic lift, then all suitable lifts of $\bar{\rho}$ are automorphic. Here we quote one such result (see [12], [15], [23, 24] and [27] for many other modularity lifting theorems):

Theorem 2.2. [8, Theorem A] *Let $\bar{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation satisfying* $(\mathbf{LI}_{\bar{\rho}})$ *and* $(\mathbf{Mod}_{\bar{\rho}})$*. Then all crystalline lifts of $\bar{\rho}$ of Hodge-Tate weights between 0 and $p-2$ which are unramified outside a finite set of primes are automorphic.*

3. Hilbert modular varieties

3.1. **Definition.** For an open compact subgroup K of $\mathrm{GL}_2(\mathbb{A}_f)$ we define the Hilbert modular variety of level K as

$$Y_K = \mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})/KK_\infty^+,$$

where $K_\infty^+ = \mathrm{SO}_2(F_\infty)F_\infty^\times$.

We define the adjoint Hilbert modular variety of level K as:

$$Y_K^{\mathrm{ad}} = \mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})/\,\mathbb{A}^\times\,KK_\infty^+.$$

By definition there is a natural homomorphism $Y_K \to Y_K^{\mathrm{ad}}$ and the latter can be rewritten in terms of the adjoint group PGL_2 as follows:

$$Y_K^{\mathrm{ad}} = \mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})/\bar{K}\cdot\mathrm{PSO}_2(F_\infty),$$

where $\bar{K}$ is the image of K in $\mathrm{PGL}_2(\mathbb{A}_f)$.

The inclusion of open compact subgroups $K' \subset K$ induces natural homomorphisms $Y_{K'} \to Y_K$ and $Y_{K'}^{\mathrm{ad}} \to Y_K^{\mathrm{ad}}$.

3.2. **Connected components.** By the strong approximation theorem for GL_2, the set $\pi_0(Y_K)$ of connected components of Y_K is isomorphic to the class group $\mathcal{C}^+_{\det(K)}$, whereas $\pi_0(Y_K^{\mathrm{ad}})$ is isomorphic to the quotient of $\mathcal{C}^+_{\det(K)}$ by the image of $\mathbb{A}^{\times 2}$, hence it is a 2-group.

For any open compact subgroups $K' \subset K$ there are exact sequences:

$$\begin{aligned}
&1 \to \frac{\det(K)}{\det(K')(\det(K)\cap F^\times F_\infty^+)} \to \pi_0(Y_{K'}) \to \pi_0(Y_K) \to 1,\\
&1 \to \frac{\det(K)}{\det(K')(\det(K)\cap \mathbb{A}^{\times 2}\,F^\times)} \to \pi_0(Y_{K'}^{\mathrm{ad}}) \to \pi_0(Y_K^{\mathrm{ad}}) \to 1.
\end{aligned} \tag{2}$$

If $\det(K) = (\mathcal{O}_F \otimes \widehat{\mathbb{Z}})^\times$ then $\pi_0(Y_K)$ is isomorphic to the *narrow class group* $\mathcal{C}_F^+$ of F, while $\pi_0(Y_K^{\mathrm{ad}})$ is isomorphic to the *genus group* $\mathcal{C}_F^+/\mathcal{C}_F^2 \simeq \mathcal{C}_F^+/(\mathcal{C}_F^+)^2$ of F.

We will now express each connected component of Y_K in more classical terms as a quotient of $G_\infty^+/K_\infty^+ \simeq \mathfrak{H}_F$ (the d-fold product of upper half planes) by a certain congruence subgroup of the Hilbert modular group.

Choose elements $g_i \in \mathrm{GL}_2(\mathbb{A}_f)$, $1 \le i \le h$, such that $(\det(g_i))_{1\le i\le h}$ forms a set of representatives of $\pi_0(Y_K) \simeq \mathcal{C}^+_{\det(K)}$. By the strong approximation theorem for GL_2, the maps $\gamma_i \in G_\infty^+ \mapsto g_i\gamma_i \in \mathrm{GL}_2(\mathbb{A})$, $1 \leq i \leq h$ induce an isomorphism:

$$\coprod_{1\leq i\leq h} \Gamma_{g_i}\backslash G_\infty^+/K_\infty^+ \simeq \coprod_{1\leq i\leq h} \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(F)g_iKG_\infty^+/KK_\infty^+ = Y_K, \tag{3}$$

where for $g \in \mathrm{GL}_2(\mathbb{A}_f)$ we denote $\Gamma_g = \mathrm{GL}_2(F) \cap gKg^{-1}G_\infty^+$.

Similarly, each connected component of Y_K^{ad} can be defined more classically using subgroups of the Hurwitz-Maass extension of the Hilbert modular group (see [29, Chap.I]). Explicitely:

$$\coprod_{1\leq i\leq h} \Gamma_{g_i}^{\mathrm{ad}}\backslash G_\infty^+/K_\infty^+ \simeq \coprod_{1\leq i\leq h} \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(F)g_iKG_\infty^+/\,\mathbb{A}^\times KK_\infty^+ = Y_K^{\mathrm{ad}},$$

where for $g \in \mathrm{GL}_2(\mathbb{A}_f)$ we denote $\Gamma_g^{\mathrm{ad}} = \mathrm{GL}_2(F) \cap \mathbb{A}^\times gKg^{-1}G_\infty^+$ and the $g_i \in \mathrm{GL}_2(\mathbb{A}_f)$, $1 \leq i \leq h$ are chosen so that $(\det(g_i))_{1\leq i\leq h}$ is a set of representatives for $\pi_0(Y_K^{\mathrm{ad}})$.

3.3. **Cusps and compactifications.** The analytic varieties Y_K and Y_K^{ad} are quasi-projective, but never projective. The minimal compactification of Y_K is defined using (3) as

$$\overline{Y}_K \simeq \coprod_{1\leq i\leq h} \Gamma_{g_i}\backslash(\mathfrak{H}_F \cup \mathbb{P}^1(F)).$$

The complement of Y_K in $\overline{Y}_K$ consists of a finite number of points (the cusps). Since $d > 1$, the cusps are always singular points. For a Hilbert modular surface ($d = 2$) the resolution of the cusp singularities was found by Hirzebruch (see [29, Chap.II]). The variety Y_K has toroidal compactifications $\widetilde{Y}_K$, depending on some combinatorial data (see [10, §3]). The varieties $\widetilde{Y}_K$ are proper and smooth at infinity (that is to say smooth if Y_K is smooth). There exists a projection $\mathrm{pr} : \widetilde{Y}_K \to \overline{Y}_K$ inducing identity on the open Y_K and such that $\mathrm{pr}^{-1}(\{\text{cusps}\})$ is a divisor with normal crossings.

3.4. **Smoothness.** The analytic varieties Y_K and Y_K^{ad} are smooth if K is *sufficiently* small in a sense that we will now make precise.

Definition 3.1. We say that K is *neat* if, for all $g \in \mathrm{GL}_2(\mathbb{A}_f)$, the quotient of the group $\Gamma_g = \mathrm{GL}_2(F) \cap gKg^{-1}G_\infty^+$ by its center $F^\times \cap gKg^{-1}F_\infty^\times$ is torsion free. Similarly, we say that $\mathbb{A}^\times K$ is neat if, for all $g \in \mathrm{GL}_2(\mathbb{A}_f)$, the group $(\mathrm{GL}_2(F) \cap \mathbb{A}^\times gKg^{-1}G_\infty^+)/F^\times$ is torsion free.

Lemma 3.2. *Let $K' \subset K$ be two open compact subgroups of* $\mathrm{GL}_2(\mathbb{A}_f)$.

(i) *The variety Y_K (resp. Y_K^{ad}) is an orbifold.*
(ii) *The variety Y_K (resp. Y_K^{ad}) is smooth if, and only if, K (resp. $\mathbb{A}^\times K$) is neat.*
(iii) *If K is neat, then K' is neat. If $\mathbb{A}^\times K$ is neat, then $\mathbb{A}^\times K'$ and K are neat too.*

Proof. (i) Recall that Y_K admits a complex uniformization as in (3). For every $g \in \mathrm{GL}_2(\mathbb{A}_f)$ the kernel of the action of Γ_g on G_∞^+/K_∞^+ is precisely given by its center $F^\times \cap KF_\infty^\times$. The latter is a subgroup of $\mathcal{O}_F^\times$ of finite index. It follows that for every $\gamma \in G_\infty^+$, locally at the point $\mathrm{GL}_2(F)g\gamma KK_\infty^+$, Y_K at is a quotient of G_∞^+/K_∞^+ (the d-fold product of upper half planes) by the group:

$$\Gamma_{g,\gamma} = \frac{\mathrm{GL}_2(F) \cap gKg^{-1}\gamma K_\infty^+\gamma^{-1}}{F^\times \cap KF_\infty^\times}. \tag{4}$$

We will now show that $\Gamma_{g,\gamma}$ is finite. The determinant maps $\Gamma_{g,\gamma}$ to

$$\frac{F^\times \cap \det(K)F_\infty^+}{(F^\times \cap KF_\infty^\times)^2}$$

which is finite as a quotient of two finite index subgroups of $\mathcal{O}_F^\times$. Finally the kernel of the determinant is generated by $\mathrm{SL}_2(F) \cap gKg^{-1}\gamma K_\infty^+\gamma^{-1}$ which is finite since $\mathrm{SL}_2(F) \subset \mathrm{SL}_2(\mathbb{A})$ is discrete while $gKg^{-1}\gamma K_\infty^+\gamma^{-1} \cap \mathrm{SL}_2(\mathbb{A})$ is compact. This shows that Y_K is an orbifold. Since Y_K^{ad} is a quotient of Y_K by the finite group $\mathcal{C}_{K\cap\mathbb{A}_f^\times}$ it is an orbifold too.

(ii) By (i), Y_K is a manifold if, and only if, $\Gamma_{g,\gamma}$ is trivial for all g and γ, which is equivalent to K being neat (one uses here that a finite order linear fractional transformation of $\mathfrak{H}_F$ has a fixed point). Similarly, Y_K^{ad} is a manifold if, and only if,

$$\Gamma_{g,\gamma}^{\mathrm{ad}} = \frac{\mathrm{GL}_2(F) \cap \mathbb{A}^\times gKg^{-1}\gamma K_\infty^+\gamma^{-1}}{F^\times}$$

is trivial for all g and γ, which is equivalent to $\mathbb{A}^\times K$ being neat.

Note that we have an exact sequence:

$$1 \to \Gamma_{g,\gamma} \to \Gamma_{g,\gamma}^{\mathrm{ad}} \to \mathcal{C}_{K\cap\mathbb{A}_f^\times},$$

where the last homomorphism in induced from:

$$uk \in \mathbb{A}^\times gKg^{-1}\gamma K_\infty^+\gamma^{-1} \mapsto u \in \frac{\mathbb{A}^\times}{\mathbb{A}^\times \cap gKg^{-1}\gamma K_\infty^+\gamma^{-1}} = \frac{\mathbb{A}^\times}{\mathbb{A}^\times \cap KF_\infty^\times}.$$

(iii) follows from the fact that for all $g \in \mathrm{GL}_2(\mathbb{A}_f)$ we have inclusions:

$$\frac{\mathrm{GL}_2(F) \cap gK'g^{-1}G_\infty^+}{F^\times \cap gK'g^{-1}F_\infty^+} \subset \frac{\mathrm{GL}_2(F) \cap gKg^{-1}G_\infty^+}{F^\times \cap gKg^{-1}F_\infty^+} \subset \frac{\mathrm{GL}_2(F) \cap \mathbb{A}^\times gKg^{-1}G_\infty^+}{F^\times}.$$

□

The following lemma can be easily deduced from [8, Lemmas 2.1, 2.2] and shows that K (resp. $\mathbb{A}^\times K$) will be neat if we carefully chose its local component at one place.

Lemma 3.3. *Let u be a prime ideal of F satisfying $\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(u) \equiv -1 \pmod{4\ell}$ for all prime numbers ℓ such that $[F(\sqrt[\ell]{1}) : F] = 2$. Suppose that $K = K_0(u) \times K^{(u)} \subset \mathrm{GL}_2(F_u) \times \mathrm{GL}_2(\mathbb{A}_f^{(u)})$, where $\mathbb{A}_f^{(u)}$ denotes the ring of finite adeles outside u.*

(i) *If the image of the uniformizer* ϖ_u *in* $\mathcal{C}_{U(4)}$ *is trivial, then* $\mathbb{A}^\times K$ *is neat.*
(ii) *If the image of the uniformizer* ϖ_u *in the* 2*-part of* $\mathcal{C}_F$ *is trivial then* K *is neat.*

3.5. **Etale coverings.**

Proposition 3.4. *Let* $K' \lhd K$ *be two open compact subgroups of* $\mathrm{GL}_2(\mathbb{A}_f)$*.*
(i) *If* Y_K *is smooth, then* $Y_{K'}$ *is smooth and the natural morphism* $Y_{K'} \to Y_K$ *is etale with group* $K/K'(K \cap F^\times F_\infty^\times)$*.*
(ii) *If* Y_K^{ad} *is smooth, then* Y_K *is smooth and the natural morphism* $Y_K \to Y_K^{\mathrm{ad}}$ *is etale with group* $\mathcal{C}_{K\cap\mathbb{A}_f^\times}$*.*
(iii) *If* Y_K^{ad} *is smooth, then* $Y_{K'}^{\mathrm{ad}}$ *is smooth and the natural morphism* $Y_{K'}^{\mathrm{ad}} \to Y_K^{\mathrm{ad}}$ *is etale with group* $K/K'(K \cap \mathbb{A}_f^\times)$*.*

Proof. (i) The group $K/K'(K \cap F^\times F_\infty^\times)$ acts on the fibers of the morphism $Y_{K'} \to Y_K$. Under the assumption that K is neat, we will show that the action is free. Suppose that $k \in K$ fixes the point $\mathrm{GL}_2(F)g\gamma K'K_\infty^+$ on $Y_{K'}$, where $g \in \mathrm{GL}_2(\mathbb{A}_f)$ and $\gamma \in G_\infty^+$, that is to say $gkg^{-1} \in \mathrm{GL}_2(F)gK'g^{-1}\gamma K_\infty^+\gamma^{-1}$. Since K is neat, the group $\Gamma_{g,\gamma}$ defined in (4) is trivial, hence:

$$gKg^{-1} \cap \mathrm{GL}_2(F)\gamma K_\infty^+\gamma^{-1} = K \cap F^\times F_\infty^\times.$$

It follows immediately that $gkg^{-1} \in gK'g^{-1}(K \cap F^\times F_\infty^\times)$ hence $k \in K'(K \cap F^\times F_\infty^\times)$ as desired.

Alternative one can reason component-wise using the fact that for $g \in \mathrm{GL}_2(\mathbb{A}_f)$ the fundamental group of $\Gamma_g\backslash\mathfrak{H}$ is $\Gamma_g/(F^\times \cap KF_\infty^\times)$. By (2) the claim would follow from the exactness of the following sequence:

$$1 \to \frac{\Gamma_g}{\Gamma'_g(F^\times \cap KF_\infty^\times)} \xrightarrow{f} \frac{K}{K'(K \cap F^\times F_\infty^\times)} \xrightarrow{\det} \frac{\det(K)}{\det(K')(\det(K) \cap F^\times F_\infty^+)} \to 1$$

where f stands for the projection onto the finite adeles followed by the conjugation by g. The surjectivity is obvious, while the injectivity follows from:

$$\mathrm{GL}_2(F) \cap \big(gK'g^{-1}G_\infty^+(F^\times \cap KF_\infty^\times)\big) = \Gamma'_g(F^\times \cap KF_\infty^\times).$$

Finally, the exactness in the middle is equivalent to $\det(f(\Gamma_g)) = \det(K) \cap F^\times F_\infty^+$. Let $k \in K$ be such that $\det(k) \in F^\times F_\infty^+$. Then

$$\mathrm{SL}_2(F) \cap \left(\begin{smallmatrix} \det(k)^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) gkg^{-1}(gKg^{-1} \cap \mathrm{SL}_2(\mathbb{A}_f))\,\mathrm{SL}_2(F_\infty) \neq \varnothing,$$

as an intersection of a dense and an open subset, hence $\det(k) \in \det(f(\Gamma_g))$.

(ii) We already mentioned that the group $\mathcal{C}_{K\cap\mathbb{A}_f^\times}$ acts on the fibers of the morphism $Y_K \to Y_K^{\mathrm{ad}}$ and we will now show that this action is free. Suppose that $u \in \mathbb{A}^\times$ fixes the point $\mathrm{GL}_2(F)g\gamma KK_\infty^+$, where $g \in \mathrm{GL}_2(\mathbb{A}_f)$ and $\gamma \in G_\infty^+$, that is to say $u \in \mathrm{GL}_2(F)gKg^{-1}\gamma K_\infty^+\gamma^{-1}$. Since $\mathbb{A}^\times K$ is neat, the proof of lemma 3.2(ii) yields

$$\mathrm{GL}_2(F) \cap \mathbb{A}^\times gKg^{-1}\gamma K_\infty^+\gamma^{-1} = F^\times,$$

hence $\mathrm{GL}_2(F)gKg^{-1}\gamma K_\infty^+\gamma^{-1} \cap \mathbb{A}^\times = F^\times(gKg^{-1}\gamma K_\infty^+\gamma^{-1} \cap \mathbb{A}^\times)$ and $u \in F^\times(KF_\infty^\times \cap \mathbb{A}^\times)$ as desired.

(iii) The group $K/K'(K \cap \mathbb{A}_f^\times)$ acts on the fibers of the morphism $Y_{K'}^{\mathrm{ad}} \to Y_K^{\mathrm{ad}}$. The freeness of this action can be established either as in (i), by showing for example the exactness of:

$$1 \to \frac{\Gamma_g^{\mathrm{ad}}}{\Gamma_g'^{\,\mathrm{ad}}} \xrightarrow{f} \frac{K}{K'(K \cap \mathbb{A}_f^\times)} \xrightarrow{\det} \frac{\det(K)}{\det(K')(\det(K) \cap \mathbb{A}^{\times 2} F^\times)} \to 1,$$

or alternatively as follows. Consider the commutative diagram:

$$\begin{tikzcd} & Y_{K'} \arrow[dl] \arrow[dr] & \\ Y_K \arrow[dr] & & Y_{K'}^{\mathrm{ad}} \arrow[dl] \\ & Y_K^{\mathrm{ad}} & \end{tikzcd}$$

Since by (i) and (ii) we already know that the other three morphisms are etale, to show that $Y_{K'}^{\mathrm{ad}} \to Y_K^{\mathrm{ad}}$ is etale of group $K/K'(K \cap \mathbb{A}_f^\times)$ it is enough to check that:

$$\left[K : K'(K \cap F^\times F_\infty^\times)\right] = \left[K : K'(K \cap \mathbb{A}_f^\times)\right] \cdot \left[\mathcal{C}_{K' \cap \mathbb{A}_f^\times} : \mathcal{C}_{K \cap \mathbb{A}_f^\times}\right],$$

which is true, since $\left[K'(K \cap \mathbb{A}_f^\times) : K'(K \cap F^\times F_\infty^\times)\right] =$

$$\left[K \cap \mathbb{A}_f^\times : (K \cap F^\times F_\infty^\times)(K' \cap \mathbb{A}_f^\times)\right] = \left[F^\times F_\infty^\times (K \cap \mathbb{A}_f^\times) : F^\times F_\infty^\times (K' \cap \mathbb{A}_f^\times)\right].$$

□

From now on we will only consider open compact subgroups K which are neat.

3.6. **Integral models.** Since Y_K and Y_K^{ad} are Shimura varieties for the algebraic groups $\mathrm{GL}_2(F)$ and $\mathrm{PGL}_2(F)$ over $\mathbb{Q}$, they have canonical models over a number field, which is $\mathbb{Q}$ if for example $K = K_0(\mathcal{N})$ or $K_1(\mathcal{N})$.

Since Y_K and Y_K^{ad} turn out to be (course) moduli spaces classifying Hilbert-Blumenthal abelian varieties with some additional structures, Mumford's Geometric Invariant Theory yields integral models which are smooth away from the discriminant of F and away from primes v where K_v is not maximal.

Finally, the $\widetilde{Y}_K$'s have smooth rational and integral models over the same base as Y_K (see [21] for $K = K(\mathcal{N})$ and [9] for $K = K_0(\mathcal{N})$, $K_1(\mathcal{N})$ and $K_{11}(\mathcal{N})$).

3.7. **Betti cohomology with p-adic coefficients.** We fix a prime p and a p-adic field E containing the Galois closure of F of $\bar{\mathbb{Q}}_p$, and denote by $\mathcal{O}$ its ring of integers. We fix an embedding of $\bar{\mathbb{Q}}$ in $\bar{\mathbb{Q}}_p$ allowing us to identify $\mathrm{GL}_2(\mathcal{O}_F \otimes \mathcal{O})$ with $\mathrm{GL}_2(\mathcal{O})^{J_F}$.

For any arithmetic weight (k, w_0) and any $\mathcal{O}$-algebra A, we consider the following algebraic representation of $\mathrm{GL}_2(\mathcal{O}_F \otimes A) \simeq \mathrm{GL}_2(A)^{J_F}$:

$$\mathcal{L}_K(k, w_0; A) := \bigotimes_{\tau \in J_F} \mathrm{Det}^{\frac{w_0 - k_\tau}{2}+1} \otimes \mathrm{Sym}^{k_\tau - 2}(A^2). \tag{5}$$

Let $\mathcal{L}_K(k, w_0; A)$ be the sheaf of locally constant sections of

$$\mathrm{GL}_2(F) \backslash (\mathrm{GL}_2(\mathbb{A}) \times \mathcal{L}(k, w_0; A)) / KK_\infty^+ \longrightarrow Y_K,$$

where the action on $\mathcal{L}(k, w_0; A)$ is via $K_p := \prod_{v|p} K_v \subset \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$.

We consider Betti cohomology groups $\mathrm{H}^\bullet(Y_K, \mathcal{L}_K(k, w_0; \mathcal{O}))$ and their compactly supported versions $\mathrm{H}_c^\bullet(Y_K, \mathcal{L}_K(k, w_0; \mathcal{O}))$. We will see in §4.1 that under certain conditions these groups will be torsion free.

3.8. **Hecke correspondances.** Note that for $K' \subset K$, there is a natural projection $\mathrm{pr} : Y_{K'} \to Y_K$ and $\mathrm{pr}^* \mathcal{L}_K(k, w_0; A) = \mathcal{L}_{K'}(k, w_0; A)$. For $g \in \mathrm{GL}_2(\mathbb{A}_f)$ we define the Hecke correspondence $[KgK]$ on Y_K by the usual diagram:

$$\begin{tikzcd} & Y_{K \cap gKg^{-1}} \arrow[dl, "\mathrm{pr}_1"'] \arrow[r, "\cdot g"] & Y_{g^{-1}Kg \cap K} \arrow[dr, "\mathrm{pr}_2"] & \\ Y_K & & & Y_K \end{tikzcd} \tag{6}$$

According to [13, §7], if $g_p \in M_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$ then $[KgK]$ induces an endomorphism of $\mathrm{H}^\bullet(Y_K, \mathcal{L}_K(k, w_0; A))$ and of $\mathrm{H}_c^\bullet(Y_K, \mathcal{L}_K(k, w_0; A))$.

If K_v is maximal, we define the standard Hecke operators $T_v = [K_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K_v] = [K_v \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) K_v]$ and $S_v = [K_v \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K_v] = [\left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K_v]$. For all other v we define the Hecke operator $U_v = [K_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K_v]$.

Similarly, we have Betti cohomology groups $\mathrm{H}^\bullet(Y_K^{\mathrm{ad}}, \mathcal{L}_K(k, w_0; A))$ and Hecke action on them. In particular, if K_v is maximal, there is a Hecke operator T_v (the operator S_v acts by $\mathrm{N}_{F/\mathbb{Q}}(v)^{w_0}$).

3.9. **Poincare duality.** In this section we will endow the middle degree cohomology of a Hilbert modular variety with various pairings coming from the Poincare duality.

We consider the dual sheaves $\mathcal{L}_\mathcal{O} = \mathcal{L}(k, w_0; \mathcal{O})$ and $\mathcal{L}_\mathcal{O}^\vee = \mathcal{L}(k, -w_0; \mathcal{O})$ on Y_K (see (5)). The cup product followed by the trace map induces a pairing:

$$[\ ,\ ] : \mathrm{H}_c^d(Y_K, \mathcal{L}_\mathcal{O}) \times \mathrm{H}^d(Y_K, \mathcal{L}_\mathcal{O}^\vee) \to \mathrm{H}_c^{2d}(Y_K, \mathcal{O}) \to \mathcal{O}, \tag{7}$$

which becomes perfect after extending scalars to E. The dual of the Hecke operator $[KgK]$ under this pairing is the Hecke operator $[Kg^{-1}K]$ (*cf* [12, §3.4]). In particular, the dual of T_v (resp. S_v) is $T_v S_v^{-1}$ (resp. S_v^{-1}) We will modify the pairing (7) in a standard way, in order to make it Hecke equivariant.

First, the involution $g \mapsto g^* = (\det g)^{-1} g$ of GL_2 induces a natural isomorphism $\mathrm{H}^d(Y_K, \mathcal{L}_\mathcal{O}^\vee) \simeq \mathrm{H}^d(Y_{K^*}, \mathcal{L}_\mathcal{O})$.

Assume next that K has level $\mathcal{N}$, an ideal of $\mathcal{O}_F$ prime to p, in the sense that $\iota K^* = K\iota$, where $\iota = \left(\begin{smallmatrix} 0 & -1 \\ \mathcal{N} & 0 \end{smallmatrix}\right)$. Then $\iota^* \mathcal{L}_{\mathcal{O}} \simeq \mathcal{L}_{\mathcal{O}}$ and there is a natural isomorphism:

$$\mathrm{H}^d(Y_{K^*}, \mathcal{L}_{\mathcal{O}}) \simeq \mathrm{H}^d(Y_{\iota K^* \iota^{-1}}, \mathcal{L}_{\mathcal{O}}) = \mathrm{H}^d(Y_K, \mathcal{L}_{\mathcal{O}}).$$

Since for all diagonal g we have $\iota g^* \iota^{-1} = g^{-1}$ the following diagram commutes:

(8)

$$\begin{tikzcd} \mathrm{H}^d(Y_K, \mathcal{L}_{\mathcal{O}}^{\vee}) \arrow[r, "*"] \arrow[d, "{[Kg^{-1}K]}"] & \mathrm{H}^d(Y_{K^*}, \mathcal{L}_{\mathcal{O}}) \arrow[r, "{[K\iota K^*]}"] \arrow[d, "{[K^*(g^{-1})^* K^*]}"] & \mathrm{H}^d(Y_{\iota K^* \iota^{-1}}, \mathcal{L}_{\mathcal{O}}) \arrow[r, equal] & \mathrm{H}^d(Y_K, \mathcal{L}_{\mathcal{O}}) \arrow[d, "{[KgK]}"] \\ \mathrm{H}^d(Y_K, \mathcal{L}_{\mathcal{O}}^{\vee}) \arrow[r, "*"] & \mathrm{H}^d(Y_{K^*}, \mathcal{L}_{\mathcal{O}}) \arrow[r, "{[K\iota K^*]}"] & \mathrm{H}^d(Y_{\iota K^* \iota^{-1}}, \mathcal{L}_{\mathcal{O}}) \arrow[r, equal] & \mathrm{H}^d(Y_K, \mathcal{L}_{\mathcal{O}}). \end{tikzcd}$$

This shows that the *modified* Poincare pairing:

$$\langle\ ,\ \rangle = [\ ,\ \ \iota \circ *] : \mathrm{H}_c^d(Y_K, \mathcal{L}_{\mathcal{O}}) \times \mathrm{H}^d(Y_K, \mathcal{L}_{\mathcal{O}}) \to \mathcal{O}, \tag{9}$$

is equivariant for all the standard Hecke operators.

The interior cohomology group $\mathrm{H}_!^d(Y_K, \mathcal{L}_{\mathcal{O}})$ is defined as the image of $\mathrm{H}_c^d(Y_K, \mathcal{L}_{\mathcal{O}})$ in $\mathrm{H}^d(Y_K, \mathcal{L}_{\mathcal{O}})$. From commutativity of the diagram:

$$\begin{tikzcd} \mathrm{H}_c^d(Y_K, \mathcal{L}_{\mathcal{O}}) \otimes \mathrm{H}_c^d(Y_K, \mathcal{L}_{\mathcal{O}}) \arrow[r] \arrow[d] & \mathrm{H}_c^d(Y_K, \mathcal{L}_{\mathcal{O}}) \otimes \mathrm{H}^d(Y_K, \mathcal{L}_{\mathcal{O}}) \arrow[d, "{\langle\ ,\ \rangle}"] \\ \mathrm{H}^d(Y_K, \mathcal{L}_{\mathcal{O}}) \otimes \mathrm{H}_c^d(Y_K, \mathcal{L}_{\mathcal{O}}) \arrow[r, "{\langle\ ,\ \rangle}"] & \mathcal{O} \end{tikzcd}$$

and from (9) we deduce a Hecke equivariant pairing:

$$\langle\ ,\ \rangle : \mathrm{H}_!^d(Y_K, \mathcal{L}_{\mathcal{O}}) \times \mathrm{H}_!^d(Y_K, \mathcal{L}_{\mathcal{O}}) \to \mathcal{O}\,. \tag{10}$$

We will see in §4.1 that under certain conditions this pairing will be perfect.

4. Cohomology of Hilbert modular varieties

Let $K = \prod_v K_v \subset \mathrm{GL}_2(\mathbb{A}_f)$ be a neat open compact subgroup such that K_v is maximal for all primes v dividing p. Fix an arithmetic weight (k, w_0) and for every $\mathcal{O}$-algebra A put $\mathcal{L}_A = \mathcal{L}(k, w_0; A)$.

Let $\bar\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation such that $(\mathbf{Mod}_{\bar\rho})$ and $(\mathbf{LI}_{\bar\rho})$ hold. Consider the maximal ideal

$$\mathfrak{m}_{\bar\rho} = (\varpi, T_v - \mathrm{tr}(\bar\rho(\mathrm{Frob}_v)), S_v - \det(\bar\rho(\mathrm{Frob}_v))\mathrm{N}_{F/\mathbb{Q}}(v)^{-1})$$

of the abstract Hecke algebra $\mathbb{T} := \mathcal{O}[T_v, S_v |\ \ K_v \text{ maximal}, v \nmid p]$.

The Betti cohomology groups $\mathrm{H}^\bullet(Y_K, \mathcal{L}_{\mathcal{O}})$ defined in §3.7 are $\mathbb{T}$-modules and we denote by $\mathrm{H}^\bullet(Y_K, \mathcal{L}_{\mathcal{O}})_{\bar\rho}$ the localization at $\mathfrak{m}_{\bar\rho}$.

4.1. **Freeness results.** Under the above assumptions the following theorem is proved in [7, Theorems 4.4, 6.6] and [8, Theorem 2.3] (see [18] for vanishing theorems for the cohomology *without* localization).

Theorem 4.1.

(i) *The* $\mathcal{O}$*-module* $\mathrm{H}^\bullet_c(Y_K, \mathcal{L}_\mathcal{O})_{\bar\rho} = \mathrm{H}^\bullet(Y_K, \mathcal{L}_\mathcal{O})_{\bar\rho} = \mathrm{H}^d(Y_K, \mathcal{L}_\mathcal{O})_{\bar\rho}$ *is free of finite rank.*

(ii) $\mathrm{H}^d(Y_K, \mathcal{L}_{E/\mathcal{O}})_{\bar\rho}$ *is a divisible* $\mathcal{O}$*-module of finite corank and the Pontryagin pairing* $\mathrm{H}^d(Y_K, \mathcal{L}_\mathcal{O})_{\bar\rho} \times \mathrm{H}^d(Y_K, \mathcal{L}_{E/\mathcal{O}})_{\bar\rho} \to E/\mathcal{O}$ *is a perfect duality.*

(iii) *The pairing (9) yields a perfect duality of free* $\mathcal{O}$*-modules:*

$$\langle\ ,\ \rangle : \mathrm{H}^d(Y_K, \mathcal{L}_\mathcal{O})_{\bar\rho} \times \mathrm{H}^d(Y_K, \mathcal{L}_\mathcal{O})_{\bar\rho} \to \mathcal{O}\,.$$

Moreover, if $K\,\mathbb{A}^\times$ *is neat then (i) and (ii) remain valid when we replace* Y_K *by* Y_K^{ad}*.*

4.2. **Results on morphisms.** Keep the assumptions from the beginning of this section.

Theorem 4.2. [8, Theorem 2.4] *Suppose given an etale morphism of smooth Hilbert modular varieties* $Y_{K'} \to Y_K$ *with group* Δ*. Assume that* Δ *is an abelian* p*-group and that* $\mathcal{O}$ *is large enough to contain the values of all its characters. Then* $\mathrm{H}^d(Y_{K'}, \mathcal{L}_\mathcal{O})_{\bar\rho}$ *is a free* $\mathcal{O}[\Delta]$*-module and there is an isomorphism of* $\mathbb{T}$*-modules:*

$$\mathrm{H}^d(Y_{K'}, \mathcal{L}_\mathcal{O})_{\bar\rho} \otimes_{\mathcal{O}[\Delta]} \mathcal{O} \simeq \mathrm{H}^d(Y_K, \mathcal{L}_\mathcal{O})_{\bar\rho}.$$

Let v be a prime not dividing p. Assume that K_v is maximal and consider the degeneracy maps $\mathrm{pr}_1, \mathrm{pr}_2 : Y_{K\cap K_0(v)} \to Y_K$ used to define the Hecke correspondence T_v in §3.8. The following theorem generalizes Ihara's lemma on the first cohomology groups of modular curves to the middle degree cohomology of Hilbert modular varieties.

Theorem 4.3. [8, Theorem 3.1] *The* $\mathbb{T}$*-linear homomorphism:*

$$\mathrm{pr}_1^* + \mathrm{pr}_2^* : \mathrm{H}^d(Y_K, \mathcal{L}_\mathcal{O})_{\bar\rho}^{\oplus 2} \to \mathrm{H}^d(Y_{K\cap K_0(v)}, \mathcal{L}_\mathcal{O})_{\bar\rho}$$

is injective with flat cokernel.

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