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Hilbert Modular Forms modulo p of partial weight one and unramifiedness of Galois representations

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Résumé: Cette thèse étudie les formes modulaires de Hilbert de poids arbitaire avec coefficients sur un corps fini de caractéristique p. En particulier, on calcule l'action des opérateurs de Hecke, y compris aux places divisant p où ils ont été construit par Emerton, Reduzzi and Xiao, sur les q-développement géometriques attachés à ces formes. Comme application nous montrons que la représentation galoisienne attachée à une forme propre cuspidale de Hilbert mod p, qui a poids parallel 1 en une place $\mathfrak{p}$ divisant p, est non-ramifiée en $\mathfrak{p}$.

Abstract: This thesis studies Hilbert modular forms of arbitrary weight with coefficients over a finite field of characteristic p. In particular, we compute the action on geometric q-expansions attached to these forms of Hecke operators, including at places dividing p as constructed by Emerton, Reduzzi and Xiao. As an application, we prove that the Galois representation attached to a Hilbert cuspidal eigenform mod p, which has parallel weight 1 at a place $\mathfrak{p}$ dividing p, is unramified at $\mathfrak{p}$.

Ai miei genitori, Maria Pia e Cosimo
e a mio fratello Alberto.

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Introduction

This thesis is divided in two parts: in Part I, we study Galois representations of the absolute Galois group $G_{\mathbb{Q}}$ with coefficients modulo p-powers which are unramified at p, whereas in Part II we study Hilbert modular forms of partial weight, posing particular attention to partial weight one Hilbert modular forms modulo p.

In Part I, we put ouserselves in the context of Serre's modularity conjecture for weight 1 forms modulo prime powers. Serre's modularity conjecture, now a theorem of Khare and Wintenberger [KW09], states that a continuous irreducible odd Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is *modular*, i.e. it arises as the reduction modulo p of the Galois representation attached to a Katz modular eigenform. In particular, Edixhoven's formulation of the weight in Serre's conjecture states that those representations that are unramified at p correspond to Katz modular forms of weight 1 with coefficients over $\overline{\mathbb{F}}_p$. This was proven by by Gross([Gro90]) in the p-distinguished case, by Coleman and Voloch ([CV92]) for $p \geq 3$ using companion forms. Wiese ([Wie14]) showed the unramifiedness at p of the representation attached to weight 1 forms without any assumptions on the prime, i.e allowing p to be 2. In Part I, we prove one side of Serre's modularity correspondence for weight 1 forms modulo prime powers.

Theorem A. *Let $p \geq 3$. Let $\mathcal{O}$ be the ring of integers in a finite extension K of $\mathbb{Q}_p$, ϖ be a uniformizer of $\mathcal{O}$ and $\mathcal{O}/\varpi = k$. Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\varpi^m\mathcal{O})$ be a continuous representation and $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k)$ its residual representation of conductor N. Suppose that $\bar{\rho}$ and ρ are such that:*

- *$\bar{\rho}$ is odd and irreducible;*
- *for all primes ℓ, either $\rho(I_\ell) \xrightarrow{\sim} \bar{\rho}(I_\ell)$ or $\dim(\rho^{I_\ell}) = \dim(\bar{\rho}^{I_\ell})$;*
- *ρ is unramified at p.*

Then ρ is modular of weight 1, i.e. there exist a modular curve X depending on $\bar{\rho}$ and a normalized eigenform $f \in \mathcal{M}_1(X, \mathcal{O}/\varpi^m\mathcal{O})$ such that ρ is equivalent to ρ_f.

This result is an application of the $R = \mathbf{T}$ theorem of Calegari and Geraghty ([CG18, Theorem 1.3]). Calegari and Specter [CS19] show the other side of the Serre's modularity correspondence, in the sense that they show that Galois representations arising from modular forms of weight 1 with coefficients modulo ϖ^m are unramified at p.

The heart of this thesis is Part II, where we study Hilbert modular forms of arbitrary weight. In the literature, most authors work with parallel weight Hilbert modular forms, whereas here we want to work in arbitrary weight. We now proceed to present the setup of Part II.

Let F be a totally real field of degree $d > 0$ and p be a rational prime. Denote by $\mathcal{O}$ the ring of integers of a finite extension of $\mathbb{Q}_p$, by ϖ a uniformizer in $\mathcal{O}$ and let $\mathbb{F} := \mathcal{O}/\varpi\mathcal{O}$. Let Σ denote the set of p-adic embeddings of F. The weights of our forms will be indexed by this set. In particular, we decompose this set as follows $\Sigma = \cup_{\mathfrak{p}|p}\Sigma_{\mathfrak{p}}$, where $\Sigma_{\mathfrak{p}}$ is the subset of embeddings inducing the place $\mathfrak{p}$. Moreover, we fix an ordering of $\Sigma_{\mathfrak{p}} = \{\tau_{\mathfrak{p},j}^{(i)} : j = 1, \ldots, f_{\mathfrak{p}} \text{ and } i = 1, \ldots e_{\mathfrak{p}}\}$ (see Notation for more details) and uniformizers $\varpi_{\mathfrak{p}}$ of $\mathcal{O}_{F,\mathfrak{p}}$. Finally, let $\mathfrak{n}$ denote an ideal of $\mathcal{O}_F$ prime to p, which will be our level.
Since we want to allow p to ramify in F, we will work with the Pappas-Rapoport model of the Hilbert modular scheme over $\mathrm{Spec}(\mathcal{O})$, which we will denote Y (see Definition 2.1.1), as constructed by Pappas and Rapoport in [PR05] and made explicit by Sasaki in [Sas19]. This scheme classifies d-dimensional abelian schemes $\pi : A \to \mathrm{Spec}(\mathcal{O})$ endowed with a prime-to-p polarization, a $\mathfrak{n}$-level structure, and a filtration of the sheaf $\pi_*\Omega^1_{A/\mathrm{Spec}(\mathcal{O})}$, which depends on the choice of ordering of Σ. The filtration is what allows us to work with primes p that ramify in F, and we will describe it here over the universal abelian variety $\pi : \mathcal{A} \to \mathrm{Y}$. One has a natural direct sum decomposition

$$\omega_{\mathcal{A}/\mathrm{Y}} := \pi_*\Omega^1_{\mathcal{A}/\mathrm{Y}} \simeq \bigoplus_{\mathfrak{p}|p}\bigoplus_{j=1}^{f_{\mathfrak{p}}} \omega_{\mathcal{A}/\mathrm{Y},\mathfrak{p},j}.$$

Then for each $\mathfrak{p}$ and $j \in \{1, \ldots f_{\mathfrak{p}}\}$, we are given a filtration of the sheaf $\omega_{\mathcal{A}/\mathrm{Y},\mathfrak{p},j}$:

$$0 = \mathcal{F}^{(0)}_{\mathfrak{p},j} \subset \mathcal{F}^{(1)}_{\mathfrak{p},j} \subset \ldots \subset \mathcal{F}^{(e_{\mathfrak{p}})}_{\mathfrak{p},j} = \omega_{\mathcal{A}/\mathrm{Y},\mathfrak{p},j} \ ,$$

by $\mathcal{O}_F$-stable $\mathcal{O}_{\mathrm{Y}}$-subbundles, such that each subquotient is a locally free $\mathcal{O}_{\mathrm{Y}}$-module of rank one, which is annihilated by the action of $\varpi_{\mathfrak{p}}$. Using this filtration and following Emerton, Reduzzi and Xiao ([ERX17a]), we are able to define line bundles

$$\dot{\omega}_{\tau^{(i)}_{\mathfrak{p},j}} := \mathcal{F}^{(i)}_{\mathfrak{p},j}/\mathcal{F}^{(i-1)}_{\mathfrak{p},j} \ ,$$

as successive quotients of the filtration. As said before, we are interested in working with arbitrary weights, and in order to do so, one has to twist the sheaves $\dot{\omega}_\tau$ by trivial line bundles coming from the de Rham cohomology:

$$\dot{\delta}_\tau := \big(\wedge^2_{\mathcal{O}_F\otimes\mathcal{O}_{\mathrm{Y}}} \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathrm{Y})\big) \otimes_{\mathcal{O}_F\otimes\mathcal{O}_{\mathrm{Y}},\tau\otimes 1} \mathcal{O}_{\mathrm{Y}} \ .$$

These line bundles are trivial over Y, but they carry a non-trivial action of the unit group $\mathrm{E} := \mathcal{O}^\times_{F,+}/(\mathcal{O}^\times_{F,\mathfrak{n}})^2$ (see section 2.2.2). In fact, since we are interested in the Galois representations attached to Hilbert modular forms, we will have to work with the Shimura variety associated to $\mathrm{Res}^F_{\mathbb{Q}}\,\mathrm{GL}_{2,F}$. The group E acts on the points of Y (see Section 2.1.3) and if the level $\mathfrak{n}$ is sufficiently divisible (see Hypothesis 4), then the group E acts properly and discontinously giving rise to an étale finite type scheme $\mathrm{Sh} := \mathrm{Y}/\mathrm{E}$ (see [RX17, Proposition 2.4]). Using the theory of descent, one can then descend the bundles $\dot{\omega}_\tau$ and $\dot{\delta}_\tau$ to line bundles ω_τ and δ_τ over Sh.
We have now all the ingredients to construct the sheaf of Hilbert modular forms for any $\mathcal{O}$-algebra R. Let $(k,\ell) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$, and we assume that $k, \ell \in \mathbb{Z}^\Sigma$ are such that $u^{k+2\ell} := \prod_{\tau\in\Sigma}\tau(u)^{k_\tau+2\ell_\tau}$ is 1 in R, for all $u \in \mathcal{O}^\times_{F,\mathfrak{n}}$. Then we define the line bundle

$$\omega^{k,\ell}_R := \bigotimes_{\tau\in\Sigma}\Big(\omega^{\otimes k_\tau}_{\tau,R} \otimes_{\mathrm{Sh}_R} \delta^{\otimes \ell_\tau}_{\tau,R}\Big)$$

and we will call *Hilbert modular forms* elements of

$$\mathcal{M}_{k,\ell}(\mathfrak{n};R) := \mathrm{H}^0(\mathrm{Sh}_R, \omega_R^{k,\ell}).$$

The assumption that $u^{k+2\ell}$ is 1 in R is necessary to ensure that we are not only working with zero global sections. Because of this condition, when working in characteristic 0, one is obliged to work with *paritious weights*, i.e. weights $k,\ell \in \mathbb{Z}^\Sigma$ such that $k_\tau + 2\ell_\tau = \mathsf{w} \in \mathbb{Z}$ for all $\tau \in \Sigma$. However, when working over $\mathbb{F}$, one can work with non-paritious weights. A concrete example of such forms can be found in the generalized partial Hasse invariants constructed by Reduzzi and Xiao in [RX17].

We will now illustrate the main results of this thesis and the methods here used. In this thesis, we are interested in computing the *geometric q-expansions* attached to Hilbert modular forms. In the literature, authors often work with adelic q-expansions, which are a more compact type of q-expansion that in the case of parallel weight forms contain all the information relative to the eigenvalues of the forms. However, when working with arbitrary weights, the adelic coefficients are not well defined (see discussion in Section 3.3.2), and therefore one is obliged to work with geometric q-expansions. In sections 2.3 and 2.4 we detail how these q-expansions are constructed starting from the cusps and Tate objects. We recall here the key steps that we take in order to construct the module of q-expansions.

Let $\mathfrak{C}$ be a fixed set of representatives of the narrow class group Cl_F^+ and we assume without loss of generality that elements $\mathfrak{c} \in \mathfrak{C}$ are coprime with p. For every $\mathfrak{c} \in \mathfrak{C}$, one can construct various cusps attached to $\mathfrak{c}$, however we will focus on the standard cusp at infinity, here denoted by $\infty(\mathfrak{c})$. As described by Dimitrov in [Dim04], fixing a smooth admissible cone decomposition of $\mathfrak{c}_+$ gives rise to a Tate object $\mathrm{Tate}_{\mathfrak{c},\mathcal{O}_F}$ defined over a scheme $S_\mathfrak{c}$, depending on the cone decomposition (see discussion before Proposition 2.3.3). In particular, one can trivialize the sheaves $\omega_{\mathrm{Tate}_{\mathfrak{c},\mathcal{O}_F}/S_\mathfrak{c}}$ and $\mathcal{H}^1_{dR}(\mathrm{Tate}_{\mathfrak{c},\mathcal{O}_F}/S_\mathfrak{c})$, giving rise to a canonical identification:

$$\dot{\omega}^{k,\ell}_{\mathrm{Tate}_{\mathfrak{c},\mathcal{O}_F}/S_\mathfrak{c}} \overset{\mathrm{can}(\mathfrak{c},\mathcal{O}_F)}{=} (\mathfrak{c}\otimes\mathcal{O})^k \otimes_\mathcal{O} (\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell \otimes_\mathcal{O} \mathcal{O}_{S_\mathfrak{c}}\ ,$$

where by $(\mathfrak{c}\otimes\mathcal{O})^k \otimes_\mathcal{O} (\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell$ we mean the free $\mathcal{O}$-module of rank 1 defined as

$$(\mathfrak{c}\otimes\mathcal{O})^k \otimes_\mathcal{O} (\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell := \bigotimes_{\tau\in\Sigma} (\mathfrak{c}\otimes\mathcal{O})_\tau^{\otimes k_\tau} \otimes_\mathcal{O} (\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O})_\tau^{\otimes \ell_\tau},$$

where $(\mathfrak{c}\otimes\mathcal{O})_\tau$ denotes the copy of $\mathcal{O}$ in $(\mathfrak{c}\otimes\mathcal{O})$ identified via the embedding τ. This description is inspired by the works of Diamond and Sasaki in [DS17]. The coefficients of our geometric q-expansions will live in the $\mathcal{O}$-module $(\mathfrak{c}\otimes\mathcal{O})^k \otimes_\mathcal{O} (\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell$, whereas the symbols q will live in $\mathcal{O}_{S_\mathfrak{c}}$. In Section 2.4, we show the following.

Theorem B. *Let $\mathfrak{c} \in \mathfrak{C}$. The the module of q-expansions for Hilbert modular forms of weight (k,ℓ) at the infinity cusp $\infty(\mathfrak{c})$ is*

$$\mathcal{M}^{k,\ell}_\infty(\mathfrak{c}) = \left\{ \sum_{\xi\in\mathfrak{c}_+\cup\{0\}} a_\xi q^\xi \ \middle|\ a_\xi \in \left(\mathfrak{c}\otimes\mathcal{O}\right)^k \otimes_\mathcal{O} \left(\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O}\right)^\ell; a_{\varepsilon\xi} = \varepsilon^{-\ell} a_\xi \text{ for all } \varepsilon \in \mathcal{O}_{F,+}^\times \right\}.$$

This description of the module of q-expansion is a generalization of the description given by Dimitrov in [Dim04]. Moreover, our description aligns with the one of Diamond and Sasaki in

[DS17], where they assume that p is unramified in F and therefore use a different model for the Hilbert modular variety. The main ingredients of the proof are the canonical identification described above, and the action of the units $\mathcal{O}_{F,+}^{\times}$ over the cusps, given by Dimitrov in [Dim04]. We finish Chapter 2 by describing how changing cusps affects the canonical identifications and the module of q expansions.

The goal of Chapter 3 is to describe the action on geometric q-expansions of the Hecke operator at a prime $\mathfrak{p}$ dividing p as defined by Emerton, Reduzzi and Xiao in [ERX17a]. The lack of a good $T_{\mathfrak{p}}$ operator was due to the fact that the projection maps from $\mathrm{Sh}_{\mathbb{F}}(\mathfrak{p})$, the Shimura variery with extra level at $\mathfrak{p}$, to $\mathrm{Sh}_{\mathbb{F}}$ are not finite flat. To overcome this issue, Emerton, Reduzzi and Xiao use the dualizing trace map to construct a properly normalized Hecke operator $T_{\mathfrak{p}}^{\circ}$. We therefore go through this construction with particular attention at what happens at the cusps and translate it to the q-expansions for Hilbert modular forms with coefficients over $R_m := \mathcal{O}/\varpi^m\mathcal{O}$. We point out a couple of technical details that are needed to achieve our goal. First of all, we will have to work with normalized diamond operators $S_{\mathfrak{p}}^{\circ}$. As explained in Section 3.1, the normalization is essential to have invertible diamond operators at primes $\mathfrak{p}$ dividing p. Secondly, the construction of the Hecke operator $T_{\mathfrak{p}}^{\circ}$ is done only for weights $k \in \mathbb{Z}^{\Sigma}$ such that for every $\mathfrak{p}|p$:

- $k_{\tau_{\mathfrak{p},j}^{(i+1)}} \geq k_{\tau_{\mathfrak{p},j}^{(i)}}$ for all $j = 1, \ldots, f_{\mathfrak{p}}$ and $i = 1, \ldots, e_{\mathfrak{p}} - 1$;
- $pk_{\tau_{\mathfrak{p},j}^{(1)}} \geq k_{\tau_{\mathfrak{p},j-1}^{(e_{\mathfrak{p}})}}$ for all $j = 1, \ldots f_{\mathfrak{p}}$.

These weights are said to live in the *minimal cone*, denoted C^{min}, defined by Diamond and Kassaei in [DK17] and [DK20]. Assuming that the weight k belongs to the minimal cone is essential in the construction of $T_{\mathfrak{p}}^{\circ}$ by Emerton, Reduzzi and Xiao (see Proposition [ERX17a, Proposition 3.11]). Using the description of the geometric q-expansion of Theorem B, we show the following.

Theorem C. *Let $(k,\ell) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}^{\Sigma}$ such that $k \in C^{\min}$ and $\prod_{\tau\in\Sigma} \tau(u)^{k_\tau+2\ell_\tau}$ is 1 in R, for all $u \in \mathcal{O}_{F,\mathfrak{n}}^{\times}$. Let $f \in \mathrm{H}^0(\mathrm{Sh}_{R_m}, \omega_{R_m}^{k,\ell})$ and let $f = \big(f_{\mathfrak{c}}\big)_{\mathfrak{c}\in\mathfrak{C}}$, where $f_{\mathfrak{c}} = \sum_{\xi\in\mathfrak{c}_+\cup\{0\}} a_\xi q^\xi$ be its geometric q-expansions at the cusp $\infty(\mathfrak{c})$. For a place $\mathfrak{p}$ of F above p, let $\alpha, \beta \in F_+$ be such that $\mathfrak{c}\mathfrak{p} = \alpha\mathfrak{c}'$ and $\mathfrak{c}\mathfrak{p}^{-1} = \beta\mathfrak{c}''$, for $\mathfrak{c}, \mathfrak{c}', \mathfrak{c}'' \in \mathfrak{C}$. Then for $\xi \in \mathfrak{c}_+$*

$$\begin{aligned} a_\xi\big((T_{\mathfrak{p}}^{\circ} f)_{\mathfrak{c}}\big) = \mathrm{Nm}(\mathfrak{p})^{-1} &\left(\prod_{\tau\in\Sigma_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{-\ell_\tau}\right) \alpha^{k+\ell} a_{\alpha^{-1}\xi}(f_{\mathfrak{c}'}) \\ &+ \left(\prod_{\tau\in\Sigma_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{k_\tau+\ell_\tau}\right) \beta^{k+\ell} a_{\beta^{-1}\xi}\big((S_{\mathfrak{p}}^{\circ} f)_{\mathfrak{c}''}\big), \end{aligned} \tag{1}$$

with $a_{\alpha^{-1}\xi} = 0$ if $\alpha^{-1}\xi \notin \mathfrak{c}'$. We recall that we denote by α^k the element $\prod_{\tau\in\Sigma} \tau(\alpha)^{k_\tau}$.

The novelty of the above description of the action of $T_{\mathfrak{p}}^{\circ}$ on geometric q-expansion lies in the fact that it is given in its full generality, without restricting to an easier case. Emerton, Reduzzi and Xiao do give briefly a descprition of the action of $T_{\mathfrak{p}}^{\circ}$ for p inert (see [ERX17a, Remark 3.14]) using the description of Katz for Hilbert modular forms, i.e. by evaluating the form f at the Tate object $\mathrm{Tate}_{\mathfrak{c},\mathcal{O}_F}$, without constructing the module of q-expansion, as here is done in Theorem B. In the proof of Theorem C, we describe how the maps in the Hecke correspondence change the cusps, and translate this change onto the modules of q-expansions, as described in Theorem B. We would like to point out that such a concrete description of the action of the $T_{\mathfrak{p}}^{\circ}$ operator on q-expansions is essential if one desires to work with arbitrary weight Hilbert modular forms. We therefore hope that our computations and methods will be useful to authors that wish to work with geometric q-expansions. We end the Chapter 3 by conjugating our formula to known cases and by discussing the adelic q-expansions, with a particular attention for the parallel weight case.

In Chapter 4, we present a direct application of our computations on geometric q-expansions in the context of the Langlands correspondence. Under the Langlands correspondence, Hilbert modular eigenforms of parallel weight one correspond to two dimensional totally odd Artin representations. In particular, the local-global compatibility ensures that these representations are unramified at all places outside the level $\mathfrak{n}$. Dimitrov and Wiese in [DW18] proved that parallel weight 1 Hilbert modular forms modulo p give rise to Galois representations that are also unramified at p. This was also proven independently by Emerton, Reduzzi and Xiao for p inert in [ERX17a]. It is predicted by the local-global compatibility in the Langlands correspondence that Hilbert modular forms of level prime to p and partial weight 1 at places corresponding to a given prime $\mathfrak{p}$ dividing p should still give rise to Galois representation which are unramified at p. In characteristic 0, this refined version of the local-global compatibility is due to Saito ([Sai09]) and Skinner ([Ski09]) (see also results Hida in [Hid89a] and Wiles in [Wil88]). In Chapter 4, we prove the analogous for Hilbert modular forms of partial weight 1 modulo p, which is not covered by the characteristic 0 case, since these forms do not lift in characteristic 0 in general. In particular, we prove the following.

Theorem D. *Let $\mathfrak{p}$ be a place above p. Let f be a Hilbert modular cuspidal eigenform of paritious weight over a finite extension of $\mathbb{F}_p$ such that the weight above $\mathfrak{p}$ is 1. Then the attached Galois representation attached to f is unramified at $\mathfrak{p}$.*

We will now explain the ingredients of the proof. In Chapter 4, we will only work with paritious weights, since we need to lift Hilbert modular forms over $\mathbb{F}$ to Hilbert modular forms over $\mathcal{O}$ for sufficently big weights. Then for $(k, \ell) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}^{\Sigma}$ a paritious weight, i.e. such that $k_\tau + 2\ell_\tau = \mathsf{w}$, we denote the sheaf of differentials of paritious weight (k, w) by

$$\omega^{(k,\mathsf{w})} := \bigotimes_{\tau\in\Sigma} \left(\omega_\tau^{\otimes k_\tau} \otimes_{\mathcal{O}_{\mathrm{Sh^{tor}}}} \delta_\tau^{\otimes(\mathsf{w}-k_\tau)/2}\right) .$$

and in particular we denote by $\mathcal{M}_{k,\mathsf{w}}(\mathfrak{n}; R) := \mathrm{H}^0\big(\mathrm{Sh}_R, \omega_R^{(k,\mathsf{w})}\big)$ the R-module of *Hilbert modular forms*, and by $\mathcal{S}_{k,\mathsf{w}}\big(\mathfrak{n}; R\big)$ the submodule of *cuspidal forms*.

In order to lift to characteristic 0, we will use an exceptional sheaf of paritious weights, denoted $(\mathrm{ex}, 0)$ such that the weight ex belongs to the minimal cone (see Section 4.1.1). The sheaf $\omega^{(\mathrm{ex},0)}$ is inspired by the one used by Reduzzi and Xiao in [RX17]. In particular, for an integer r sufficiently big, we will able to lift cuspidal forms of paritious weight $(k+r\,\mathrm{ex}, \mathsf{w})$ to characteristic 0 (see Lemma 4.1.6). Moreover, we will make use of the partial Hasse invariants defined by Reduzzi and Xiao in [RX17] to construct a Hilbert modular form $h_{\mathrm{ex}} \in \mathcal{M}_{(p-1)\,\mathrm{ex},0}(\mathfrak{n}; \mathbb{F})$ (see Lemma 4.2.2), which will allow us to bring forms to liftable weight. The final ingredient will be a *Frobenius operator* at $\mathfrak{p}$, constructed using the Hecke operator $T_{\mathfrak{p}}$ and the product of partial Hasse invariants h_{ex}. The proof then follows from the doubling method as applied by Dimitrov and Wiese in [DW18], which relies on the explicit description of the action of $T_{\mathfrak{p}}^{\circ}$ in geometric q-expansions as described in Theorem C. We end the Chapter by discussing a possible future application of these results to show the unramifiedness of the pseudo-representations attached to Hecke algebra of paritious weights (k, w) such that $k_\tau = 1$ for all $\tau \in \Sigma_{\mathfrak{p}}$.

Part I

Modular forms of weight one and Galois representations modulo prime powers

Chapter 1

Modular forms of weight one and Galois representations modulo prime powers

1.1 Introduction

Let $p \geq 3$ be a prime number and $\mathcal{O}$ the valuation ring in a finite extension K of $\mathbb{Q}_p$. Let ϖ be a uniformizer and $k = \mathcal{O}/\varpi$ the residue field. The question we want to answer is the following: given a representation

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathcal{O}/\varpi^m\mathcal{O}) \ ,$$

when is this representation modular of weight 1?

This question is part of the much larger picture of Serre's modularity conjectures for weight 1 forms. Edixhoven's formulation of the weight in Serre's conjecture ([Edi92]) states that a continuous irreducible odd Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ that is unramified at p corresponds to a Katz modular forms of weight 1 with coefficients over $\overline{\mathbb{F}}_p$. Nowadays this is entirely known. This was proven by Gross([Gro90]) in the p-distinguished case and by Coleman and Voloch ([CV92]) for $p \geq 3$ using companion forms. Wiese ([Wie14]) showed the unramifiedness at p of the representation attached to weight 1 forms without any assumptions on the prime, i.e allowing p to be 2. A sketch of the proof for the converse for $p = 2$ can be found in [Per]. A proof of the modularity of an irreducible continuous odd Galois representation with coefficients over $\overline{\mathbb{F}}_p$ was given by Khare and Wintenberger [KW09].

The first step in answering the above question is to define the space of modular forms that will correspond to ρ. In order to do so, one has to consider a modular curve X over $\mathrm{Spec}(\mathcal{O})$, which depends on the representation ρ and construct Katz modular forms of weight 1 with coefficients modulo $\varpi^m\mathcal{O}$. We will here denote this space $\mathcal{M}_1(X, \mathcal{O}/\varpi^m\mathcal{O})$ (For a precise definition see Definition 1.3.1 or Definition 1.4.2). It is important to remark that this construction depends on the ramification of the residual representation $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k)$. In particular, one has to be careful with the set $T(\bar{\rho})$ of so called *vexing primes*, which are defined at the beginning of section 1.2. Let us assume the following.

Hypothesis 1. Assume that the given continuous Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\varpi^m\mathcal{O})$ and its residual representation $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k)$ satisfy the following:

- $\bar{\rho}$ is odd and irreducible;
- for all primes ℓ such that $\bar{\rho}$ is unramified and for all primes $\ell \in T(\bar{\rho})$, $\rho(I_\ell) \xrightarrow{\sim} \bar{\rho}(I_\ell)$;

- for all ramification primes ℓ that are not vexing and for which $\bar{\rho}_{|G_\ell}$ is reducible, ρ^{I_ℓ} is a rank 1 direct summand of ρ as an $\mathcal{O}$-module;
- ρ is unramified at p.

The second step in answering this question is to say what it means for such representations to be modular of weight 1. Given a normalized eigenform $f \in \mathcal{M}_1(X, \mathcal{O}/\varpi^m\mathcal{O})$, Carayol [Car94] shows that one can construct a Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\varpi^m\mathcal{O})$, which is in particular unramified for primes away from p and from the level of f. Saying that the representation ρ is modular of weight 1 means that there exists a normalized eigenform $f \in \mathcal{M}_1(X, \mathcal{O}/\varpi^m\mathcal{O})$ such that the traces of ρ_f and ρ on almost all Frobenius elements coincide. In this sense we will also say that ρ and ρ_f are equivalent.

The goal of this chapter is to show the following:

Theorem 1.1.1. *Let $p \geq 3$. Let $\mathcal{O}$ be the ring of integers in a finite extension K of $\mathbb{Q}_p$, ϖ be a uniformizer of $\mathcal{O}$ and $\mathcal{O}/\varpi = k$. Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\varpi^m\mathcal{O})$ be a continuous representation and $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k)$ its residual representation of conductor N. Suppose that $\bar{\rho}$ and ρ satisfy Hypothesis 1. Then ρ is modular of weight 1, i.e. there exist a modular curve X depending on $\bar{\rho}$ and a normalized eigenform $f \in \mathcal{M}_1(X, \mathcal{O}/\varpi^m\mathcal{O})$ such that ρ is equivalent to ρ_f.*

We should point out that the main hypothesis is that p is not a ramification prime for ρ and consequently for $\bar{\rho}$. Moreover, by the results of [KW09], we do not have to assume that the residual representation is modular. Finally, this result is an application of the $R = T$ theorem of Calegari and Geraghty in their article *Modular Lifting beyond the Taylor-Wiles Methods*, [CG18].

The converse of this problem is the following: given a modular form of weight 1 with coefficients over $\mathcal{O}/\varpi^m\mathcal{O}$, is the attached Galois representation unramified at p? This question is answered by Calegari and Specter [CS19], who show that Galois representations arising from modular forms of weight 1 are unramified at p.

1.2 Minimal Deformations

Let $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathcal{O}/\varpi^m\mathcal{O})$ be a Galois representation and $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(k)$ its residual representation, $S(\bar{\rho})$ the set of primes at which $\bar{\rho}$ is ramified. Following Diamond (see [Dia97, Section 2]), one defines the set of vexing primes $T(\bar{\rho})$ as the subset of $S(\bar{\rho})$ of primes ℓ such that $\ell \equiv -1 \mod p$, $\bar{\rho}_{|G_\ell}$ is irreducible and $\bar{\rho}_{|I_\ell}$ is reducible. As in [CG18], let us suppose that the residual representation $\bar{\rho}$ of $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathcal{O}/\varpi^m\mathcal{O})$ satisfies the following conditions:

1. $\bar{\rho}$ is continuous, odd and absolutely irreducible;
2. $p \notin S(\bar{\rho})$;
3. If $\ell \in S(\bar{\rho})$ and $\bar{\rho}_{|G_\ell}$ is reducible, then $\bar{\rho}^{I_\ell} \neq (0)$.

Remark 1.2.1. Remark that condition 3. is always satisfied by a twist of $\bar{\rho}$ by a character unramified outside of $S(\bar{\rho})$. Moreover, for these primes the rank of $\bar{\rho}^{I_\ell}$ is necessarily 1 and therefore ℓ appears with a power ℓ^1 in the conductor of $\bar{\rho}$.

Let us recall the definition of *minimal deformation* given in [CG18]. Let $\mathcal{C}_{\mathcal{O}}$ be the category of complete Noetherian local $\mathcal{O}$-algebras with residue field k with continuous $\mathcal{O}$-algebra homomorphisms. We will consider deformations of $\bar{\rho}$ with coefficient rings in $\mathcal{C}_{\mathcal{O}}$.

Definition 1.2.2. Let R be an object of $\mathcal{C}_{\mathcal{O}}$. A deformation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(R)$ of $\bar{\rho}$ is called *minimal* if it satisfies the following conditions:

(a) the determinant $\det\rho$ is the Teichmüller lift of $\det\bar{\rho}$;

(b) for $\ell \notin S(\bar{\rho})$, $\rho_{|G_\ell}$ is unramified;

(c) for $\ell \in T(\bar{\rho})$, $\rho(I_\ell) \xrightarrow{\sim} \bar{\rho}(I_\ell)$;

(d) if $\ell \in S(\bar{\rho}) \setminus T(\bar{\rho})$ and $\bar{\rho}_{|G_\ell}$ is reducible, then ρ^{I_ℓ} is a rank one direct summand of ρ as an R-module.

Remark 1.2.3. Remark that condition (b) implies that ρ is unramified at p. Moreover, condition (d) tells us that the representation ρ as a lift of $\bar{\rho}$ not only maintains the same ramification primes, but also the same inertia invariants at those primes.

This defines a deformation problem which is representable[1] by a complete Noetherian local $\mathcal{O}$-algebra, denoted $R^{\min}$ and called the *universal minimal deformation ring.*

One can easily check that, in the setting of Theorem 1.1.1, Hypothesis 1 implies that $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\varpi^m\mathcal{O})$ is a minimal deformation of its residual representation $\bar{\rho}$.

We will now distinguish two cases:

(Case I) : The representation $\bar{\rho}$ has no vexing primes;

(Case II) : The representation $\bar{\rho}$ can have vexing primes.

We will see that the second case requires an automorphic approach, passing by the local Langlands correspondence.

1.3 Case I: No Vexing Primes

Throughout this section we will make the following assumption:

Hypothesis 2. The set of vexing primes $T(\bar{\rho}) = \emptyset$.

Following [CG18], most definitions in this section are given for a general modular curve satisfying a moduli problem. In the presence of vexing prime the considered modular curve will be a quotient of the standard modular curve $X_1(N)$ for the modular group $\Gamma_1(N)$ to include the restrictions arising from these vexing primes. Moreover, one will have to change also the sheaf of definition of modular forms, but this can be done so that in the case where Hypothesis 2 holds, one still gets the same definitions as in this section.

1.3.1 Modular Curves

Let N be an integer, $N \geq 5$ such that $(N, p) = 1$. This will later be the conductor of our representation $\bar{\rho}$. Following [CG18], fix H to be the p-part of $(\mathbb{Z}/N\mathbb{Z})^\times$. The quotient X of the modular curve $X_1(N)$ over $\mathrm{Spec}(\mathcal{O})$ by the action of H is the moduli space of generalized elliptic curves with $\Gamma_H(N)$-level structure, where $\Gamma_H(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \text{ such that } d \bmod N \in H \right\}$.

[1]This follows from Theorem 2.41 of [DDT97].

Let $\pi : \mathcal{E} \to X$ denote the universal generalized elliptic curve and set

$$\omega := \pi_* \omega_{\mathcal{E}/X} \ ,$$

where $\omega_{\mathcal{E}/X}$ is the relative dualizing sheaf. Recall that the Kodaira-Spencer map (see [Kat77], A1.3.17) extends to an isomorphism $\omega^2 \simeq \Omega^1_{X/\mathcal{O}}(\infty)$, where ∞ is the reduced divisor supported on the cusps. Let A be an $\mathcal{O}$-module and $\mathcal{L}$ a coherent sheaf on X, then we denote by[2] $\mathcal{L}_A := \mathcal{L} \otimes_{\mathcal{O}} A$.

Definition 1.3.1. Let A be an $\mathcal{O}$-module. We will call *modular forms of weight* 1 *with coefficients in* A elements of $\mathrm{H}^0(X, \omega_A)$. We will denote this module $\mathcal{M}_1(X, A)$.

In Section 3.2.2. of [CG18], Calegari and Geraghty consider $\omega_{K/\mathcal{O}}$, which can be identified with the direct limit $\varinjlim_m \omega_{\mathcal{O}/\varpi^m}$. Here we will pass through the sheaf $\omega_{K/\mathcal{O}}$ to get information on $\omega_{\mathcal{O}/\varpi^m}$. In particular, one has that $\mathrm{H}^0(X, \omega_{\mathcal{O}/\varpi^m}) \simeq \mathrm{H}^0(X, \omega_{K/\mathcal{O}})[\varpi^m]$, where this last module denotes the kernel of the morphism of sheaves 'multiplication by ϖ^m'.

1.3.2 The Hecke Algebras

In [CG18], Calegari and Geraghty define Hecke operators T_ℓ for ℓ prime such that $(\ell, Np) = 1$ and diamond operators $\langle a \rangle$ for a an integer with $(a, N) = 1$ on the cohomology $\mathrm{H}^i(X, \mathcal{L}_A)$, for $i = 0, 1$ and A any $\mathcal{O}$-module (one generally takes $\mathcal{L}$ to be the sheaf ω^n or $\omega^n(-\infty)$, for some $n \geq 1$). To do so, one considers the universal Hecke algebra, $\mathbf{T}^{\mathrm{univ}}$, which is the commutative polynomial algebra over $\mathcal{O}[(\mathbb{Z}/N\mathbb{Z})^\times]$ with indeterminates T_ℓ for ℓ prime such that $(\ell, Np) = 1$. If $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, we denote by $\langle a \rangle$ the corresponding element in $\mathbf{T}^{\mathrm{univ}}$. Then one defines an action of $\mathbf{T}^{\mathrm{univ}}$ on $\mathrm{H}^0(X, \mathcal{L}_A)$. Let $\mathbf{T}_\emptyset \subset \mathrm{End}_{\mathcal{O}} \mathrm{H}^0(X, \omega_{K/\mathcal{O}})$ be generated by the prime-to-pN Hecke operators and the prime-to-N diamond operators. Let $\mathfrak{m}_\emptyset$ be a maximal non-Eisenstein ideal of the Hecke algebra $\mathbf{T}_\emptyset$. This ideal gives rise [3] to a maximal ideal $\mathfrak{m}$ of $\mathbf{T}^{\mathrm{univ}}$ and one can assume, extending $\mathcal{O}$ if necessary, that $\mathbf{T}^{\mathrm{univ}}/\mathfrak{m} \simeq k$. The following is a particular case of part (2) of Lemma 3.7 in [CG18].

Proposition 1.3.2. *For* $i = 0, 1$*, there is an isomorphism*

$$\mathrm{H}^i(X, \omega(-\infty)_{K/\mathcal{O}})_{\mathfrak{m}} \xrightarrow{\sim} \mathrm{H}^i(X, \omega_{K/\mathcal{O}})_{\mathfrak{m}} \ .$$

1.3.3 Homology and Verdier Duality

We recall that given a profinite $\mathcal{O}$-module or a discrete torsion $\mathcal{O}$-module M, one defines the Pontryagin dual by

$$M^\vee := \mathrm{Hom}_{\mathcal{O}}(M, K/\mathcal{O}) \ .$$

Moreover, for these modules one has that $(M^\vee)^\vee \simeq M$. Now, following [CG18], one defines homology groups of modular forms as follows.

Definition 1.3.3. Let X be a modular curve and $\mathcal{L}$ be a vector bundle on X, one sets for $i = 0, 1$

$$\mathrm{H}_i(X, \mathcal{L}) := \mathrm{H}^i\big(X, (\Omega^1 \otimes \mathcal{L}^*)_{K/\mathcal{O}}\big)^\vee \ ,$$

where $\mathcal{L}^*$ is the dual bundle and $\Omega^1 \simeq \omega^{\otimes 2}(-\infty)$.[4]

[2] On open sets, $\mathcal{L}_A = \mathcal{L} \otimes_{\mathcal{O}} A$ corresponds to the sheaf tensor product over $\mathcal{O}_{X_\star}$ of $\mathcal{L}$ and $\widetilde{A}$, the sheafification of A pulled back on $X_\star$.

[3] see discussion before Lemma 3.7 in [CG18]

[4] this is induced by the Kodaira-Spencer map.

In this section, we are interested in the case where X is the modular curve defined in the previous section and $\mathcal{L}$ is just ω. The Hecke algebra $\mathbf{T}_\emptyset \subset \mathrm{End}_\mathcal{O}\,\mathrm{H}^0(X, \omega_{K/\mathcal{O}}) = \mathrm{End}_\mathcal{O}(\mathcal{M}_1(X, K/\mathcal{O}))$, defined above, acts also on $\mathrm{H}_0(X, \omega)$. In fact, by the above definition

$$\mathrm{H}_0(X,\omega) = \mathrm{Hom}_\mathcal{O}\left(\mathrm{H}^0\big(X, (\Omega^1 \otimes \omega^*)_{K/\mathcal{O}}\big), K/\mathcal{O}\right),$$

and, by looking at the sheaf $\Omega^1 \otimes \omega^*$, using Kodaira-Spencer and the fact that ω is an invertible sheaf, one gets

$$\begin{aligned}\Omega^1 \otimes \omega^* &\simeq \omega^2(-\infty) \otimes \omega^{-1} \\ &\simeq \omega(-\infty) = \omega \otimes \mathcal{L}_\infty^{-1},\end{aligned}$$

where $\mathcal{L}_\infty$ denotes the invertible sheaf associated to the divisor ∞. When tensoring with $K/\mathcal{O}$, one gets that $(\omega \otimes \mathcal{L}_\infty^{-1})_{K/\mathcal{O}} = \omega_{K/\mathcal{O}} \otimes \mathcal{L}_\infty^{-1}$. Therefore one has

$$\mathrm{Hom}_\mathcal{O}\left(\mathrm{H}^0\big(X, \omega(-\infty)_{K/\mathcal{O}}\big), K/\mathcal{O}\right) = \mathrm{H}_0(X,\omega).$$

By Proposition 1.3.2, when we localize at the maximal ideal $\mathfrak{m}_\emptyset$, one has that the homology $\mathrm{H}_0(X,\omega)_{\mathfrak{m}_\emptyset}$ is the actual dual of the cohomology $\mathrm{H}^0(X, \omega_{K/\mathcal{O}})_{\mathfrak{m}_\emptyset}$, so one still has an action of $\mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset}$ on $\mathrm{H}_0(X,\omega)_{\mathfrak{m}_\emptyset}$. Finally, by a theorem of [CG18], which will be recalled in the next section, the Hecke algebra $\mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset}$ acts freely on $\mathrm{H}_0(X,\omega)_{\mathfrak{m}_\emptyset}$.

Verdier duality ([Har66], Cor.11.2(f)) establishes an isomorphism

$$D : \mathrm{H}_i(X, \mathcal{L}) \xrightarrow{\sim} \mathrm{H}^{1-i}(X, \mathcal{L}),$$

which is not Hecke-equivariant when $\mathcal{L}$ is either $\omega^{\otimes n}$ or $\omega^{\otimes n}(-\infty)$. In fact, one gets the following relations involving the so called transposed Hecke operators T'_ℓ:

- for primes ℓ such that $(\ell, pN) = 1$, $D \circ T_\ell = T'_\ell \circ D$;
- for integers $(a, N) = 1$, $D \circ \langle a \rangle = \langle a \rangle^{-1} \circ D$.

Let us suppose that $\mathcal{O}$ contains a primitive N-th root of unity ξ, we have the extra operator W_ξ for which the transposed Hecke operators are conjugated by W_ξ to the 'usual' Hecke operators, therefore the Hecke algebras generated by these operators are the isomorphic.

1.3.4 Results of Calegari and Geraghty in [CG18]

Let X be the modular curve of level $\Gamma_H(N)$ for $N = N(\bar\rho)$ the Artin conductor of $\bar\rho$ defined above, $\mathbf{T}_\emptyset$ the Hecke algebra on $\mathrm{H}^0(X, \omega_{K/\mathcal{O}})$ generated by the prime-to-pN operators. As in [CG18], let $\mathfrak{m}_\emptyset$ be the maximal ideal[5] corresponding to $\bar\rho$. This ideal is generated by ϖ, by $T_\ell - \mathrm{tr}(\bar\rho(\mathrm{Frob}_\ell))$ for primes $(\ell, pN) = 1$, and by $\langle a \rangle - \det(\bar\rho(\mathrm{Frob}_a))$ for integers $(a, N) = 1$.

Theorem 1.3.4 (Theorem 3.11 of [CG18] for the set $Q = \emptyset$[6])**.** *There exists a minimal deformation*

$$\rho_\emptyset : G_\mathbb{Q} \longrightarrow \mathrm{GL}_2(\mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset})$$

[5]This exists thanks to the works in [KW09], [Gro90] and [CV92].

[6]The reader will notice the absence of the twist by η, where $\eta^2 = \frac{1}{\det(\rho_\emptyset)}\langle\det(\bar\rho)\rangle$ and $\langle\det(\bar\rho)\rangle$ is the Teichmüller lift of $\det(\bar\rho)$. Going through the proof, one sees that η^2 is unramified outside Q and of p-power order, therefore for $Q = \emptyset$, this twist is trivial.

of $\bar{\rho}$ unramified outside N and determined by the fact that for all primes ℓ such that $(\ell, Np) = 1$, $\mathrm{tr}(\rho_\emptyset(\mathrm{Frob}_\ell)) = T_\ell$.

The minimal deformation $\rho_\emptyset$ induces a surjective $\mathcal{C}_\mathcal{O}$-morphism

$$\varphi : R^{\min} \twoheadrightarrow \mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset} \ .$$

Using their patching method, Calegari and Geraghty show the following:

Theorem 1.3.5 (Theorem 3.25 in [CG18].). *The map $\varphi : R^{min} \to \mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset}$ obtained by the universal property of R^{min} is an isomorphism. Moreover, $\mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset}$ acts freely on $\mathrm{H}_0(X,\omega)_{\mathfrak{m}_\emptyset}$.*

From its proof (see discussion at the end of Section 3.8 of [CG18]) they deduce:

Corollary 1.3.6. *$\mathrm{H}_0(X,\omega)_{\mathfrak{m}_\emptyset}$ has rank 1 as a $\mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset}$-module.*

1.3.5 Proof of Theorem 1.1.1

The goal of this section is to prove Theorem 1.1.1 under Hypothesis 2. First we present a commutative algebra lemma that will be used to prove a q-expansion principle for these forms.

Lemma 1.3.7. *Let M be a discrete torsion $\mathcal{O}$-module. Then*

$$M^\vee/\varpi^m \simeq (M[\varpi^m])^\vee \ .$$

Proof. It suffices to show that if N is a profinite $\mathcal{O}$-module, then one has that

$$N^\vee[\varpi^m] \simeq (N/\varpi^m)^\vee \ ,$$

because then if we take N to be $M^\vee$, using the above equation one gets that

$$M[\varpi^m] = N^\vee[\varpi^m] \simeq (N/\varpi^m)^\vee = (M^\vee/\varpi^m)^\vee$$

and dualizing again will give the result. Let us now prove that $N^\vee[\varpi^m] \simeq (N/\varpi^m)^\vee$. By definition:

$$\begin{aligned} N^\vee[\varpi^m] &= \mathrm{Hom}_\mathcal{O}(N, K/\mathcal{O})[\varpi^m] \\ &= \{f : N \to K/\mathcal{O} \text{ such that } \varpi^m \cdot f = 0\} \\ &= \{f : N \to K/\mathcal{O} \text{ such that } f(\varpi^m x) = 0 \text{ for all } x \in N\} \ . \end{aligned}$$

Any such f is trivial on $\varpi^m N$ and therefore factors through the quotient $N/\varpi^m N$, defining an $\mathcal{O}$-morphism $\bar{f} : N/\varpi^m N \to K/\mathcal{O}$, i.e. an element of $(N/\varpi^m)^\vee$. The converse is obvious, so that the last term in the equality is $(N/\varpi^m)^\vee$. □

Now, we take X_U in the theorem to be just the modular curve X defined above, and $\mathcal{L}_\sigma = \mathcal{O}_X$. We can then consider $\mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset}$ the Hecke algebra acting on $\mathrm{H}^0(X,\omega_{K/\mathcal{O}})_{\mathfrak{m}_\emptyset} = \mathcal{M}_1(X, K/\mathcal{O})_{\mathfrak{m}_\emptyset}$ generated by prime-to-p Hecke operators.

Lemma 1.3.8. *We have an analogue of the q-expansion principle, i.e. a perfect pairing*

$$\mathrm{H}^0(X,\omega_{\mathcal{O}/\varpi^m\mathcal{O}})_{\mathfrak{m}_\emptyset} \times \mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset}/\varpi^m \longrightarrow \mathcal{O}/\varpi^m\mathcal{O} \ .$$

Proof. By Corollary 1.3.6, we know that $\mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset}$ and $\mathrm{H}_0(X,\omega)_{\mathfrak{m}_\emptyset} = \left(\mathrm{H}^0(X,\omega_{K/\mathcal{O}})_{\mathfrak{m}_\emptyset}\right)^\vee$ (this equality is by definition) are isomorphic as $\mathcal{O}$-modules. By Pontryagin duality, we get a perfect pairing of $\mathcal{O}$-modules:

$$\mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset} \times \mathrm{H}^0(X,\omega_{K/\mathcal{O}})_{\mathfrak{m}_\emptyset} \longrightarrow K/\mathcal{O} .$$

We want to show that we can get a perfect pairing

$$\mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset}/\varpi^m \times \mathrm{H}^0(X,\omega_{K/\mathcal{O}})_{\mathfrak{m}_\emptyset}[\varpi^n] \longrightarrow K/\mathcal{O}[\varpi^m] = \frac{1}{\varpi^m}\mathcal{O}/\mathcal{O} \simeq \mathcal{O}/\varpi^m\mathcal{O} .$$

Since $\mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset} \simeq \left(\mathrm{H}^0(X,\omega_{K/\mathcal{O}})_{\mathfrak{m}_\emptyset}\right)^\vee$, one has to show that we get an isomorphism

$$\left(\mathrm{H}^0(X,\omega_{K/\mathcal{O}})_{\mathfrak{m}_\emptyset}\right)^\vee/\varpi^m \simeq \left(\mathrm{H}^0(X,\omega_{K/\mathcal{O}})_{\mathfrak{m}_\emptyset}[\varpi^m]\right)^\vee ,$$

which is true by Lemma 1.3.7 for $M = \mathrm{H}^0(X,\omega_{K/\mathcal{O}})_{\mathfrak{m}_\emptyset}$. Now one easily sees that

$$(M[\varpi^m])^\vee = \mathrm{Hom}_\mathcal{O}(M[\varpi^m], K/\mathcal{O}[\varpi^m])$$

and one can conclude using the fact that $\mathrm{H}^0(X,\omega_{K/\mathcal{O}})[\varpi^m] = \mathrm{H}^0(X,\omega_{\mathcal{O}/\varpi^m\mathcal{O}})$. □

Remark 1.3.9. By the previous Lemma, a morphism of $\mathcal{O}$-modules

$$\mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset}/\varpi^m \to \mathcal{O}/\varpi^m\mathcal{O}$$

corresponds to a simultaneous normalized Hecke eigenvector in $\mathrm{H}^0(X,\omega_{\mathcal{O}/\varpi^m})$, thus to a normalized eigenform.

Now we can present a proof of Theorem 1.1.1 under Hypothesis 2. Let $R^{\min}$ be the minimal universal deformation ring for $\bar{\rho}$. Then applying the universal property to ρ, one gets a morphism in $\mathcal{C}_\mathcal{O}$

$$R^{\min} \longrightarrow \mathcal{O}/\varpi^m\mathcal{O} .$$

By composing with the inverse of the isomorphism of Theorem 1.3.5, we get a $\mathcal{C}_\mathcal{O}$-morphism

$$\mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset} \longrightarrow \mathcal{O}/\varpi^m\mathcal{O} ,$$

which is determined by $T_\ell \mapsto \mathrm{tr}(\rho(\mathrm{Frob}_\ell))$, by Theorem 1.3.4. One can factor this homomorphism through the quotient $\mathbf{T}_{\emptyset,\mathfrak{m}_\emptyset}/\varpi^m$ and by the previous lemma, this morphism defines a normalized eigenform f in $\mathrm{H}^0(X,\omega_{\mathcal{O}/\varpi^m\mathcal{O}})_{\mathfrak{m}_\emptyset}$ of weight 1 with coefficients in $\mathcal{O}/\varpi^m\mathcal{O}$. By the universal property of the universal deformation ring, the eigenform f corresponds to a minimal deformation ρ_f of $\bar{\rho}$ with the above conditions on images of Frobenius elements. By Chebotarev's theorem the set of Frobenii of unramified primes is dense in $G_\mathbb{Q}$, therefore $\rho \sim \rho_f$.

1.4 Case II: Vexing Primes

As Calegari and Geraghty explain, the problem that arises when the set of vexing primes is not empty is that to realize $\bar{\rho}$ by a modular form, one has to cut out a smaller space of modular forms using the local Langlands correspondence. We recall here how Calegari and Geraghty do so.

1.4.1 Modular Curves

Let $S(\bar\rho)$ and $T(\bar\rho)$ be respectively the set of ramification primes and the set of vexing primes for $\bar\rho$, as above. We set

$$P(\bar\rho) := \{\ell \in S(\bar\rho) \setminus T(\bar\rho) \text{ such that } \bar\rho_{|G_\ell} \text{ is reducible}\} .$$

For each prime $\ell \in S(\bar\rho)$, let c_ℓ denote the Artin exponent of $\bar\rho_{|G_\ell}$, i.e. $N(\bar\rho) = \prod_{\ell \neq p} \ell^{c_\ell}$. Note that c_ℓ is even for $\ell \in T(\bar\rho)$. We define local subgroups $U_\ell, V_\ell \subset \mathrm{GL}_2(\mathbb{Z}_\ell)$.

- If $\ell \in P(\bar\rho)$, we set

$$U_\ell = V_\ell = \left\{ g \in \mathrm{GL}_2(\mathbb{Z}_\ell) : g \equiv \begin{pmatrix} * & * \\ 0 & d \end{pmatrix} \mathrm{mod}(\ell^{c_\ell}), \text{ where } d \in (\mathbb{Z}/\ell^{c_\ell}\mathbb{Z})^\times \text{has } p\text{-power order} \right\} .$$

- If $\ell \in T(\bar\rho)$, let $U_\ell = \mathrm{GL}_2(\mathbb{Z}_\ell)$ and

$$V_\ell = \ker \Big(\mathrm{GL}_2(\mathbb{Z}_\ell) \longrightarrow \mathrm{GL}_2 \left(\mathbb{Z}/\ell^{\frac{c_\ell}{2}}\mathbb{Z}\right)\Big) .$$

- If $\ell \in S(\bar\rho) \setminus \big(T(\bar\rho) \cup P(\bar\rho)\big)$ then set

$$U_\ell = V_\ell = \left\{ g \in \mathrm{GL}_2(\mathbb{Z}_\ell) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mathrm{mod}(\ell^{c_\ell}) \right\} .$$

- If $\ell \notin S(\bar\rho)$, then set $U_\ell = V_\ell = \mathrm{GL}_2(\mathbb{Z}_\ell)$.

Now set

$$U = \prod_\ell U_\ell \qquad \text{and} \qquad V = \prod_\ell V_\ell .$$

Let us point out that these groups depend not only on N, but really on the behaviour of the ramification primes of $\bar\rho$. Let $\star$ be either U or V. We let $X_\star$ denote respectively the smooth projective modular curve over $\mathrm{Spec}(\mathcal{O})$ which is the moduli space of generalized elliptic curve with level $\star$ structure.[7] These curves are quotients of the usual modular curve $X_1(N)$ for the modular group $\Gamma_1(N)$. Let $\pi : \mathcal{E} \to X_\star$ be the universal generalized elliptic curve and set

$$\omega = \pi_* \omega_{\mathcal{E}/X_\star} ,$$

where $\omega_{\mathcal{E}/X_\star}$ is the relative dualizing sheaf. Let ∞ denote the reduced divisor supported on cusps. If M is an $\mathcal{O}$-module and $\mathcal{L}$ is a sheaf of $\mathcal{O}$-modules, we denote $\mathcal{L}_M$ the sheaf $\mathcal{L} \otimes_{\mathcal{O}} M$ on $X_\star$. There is a natural action of $G = U/V = \prod_{\ell \in T(\bar\rho)} \mathrm{GL}_2(\mathbb{Z}/\ell^{c_\ell/2}\mathbb{Z})$ on X_V, which gives an isomophism $X_V/G \xrightarrow{\sim} X_U$ (see Section IV of [DR73].). Let σ_ℓ be the representation of $\mathrm{GL}_2(\mathbb{Z}/\ell^{c_\ell/2}\mathbb{Z})$ with image in an $\mathcal{O}$-module W_{σ_ℓ} as in Section 5 of [CDT99]. Then $\sigma = (\sigma_\ell)_{\ell \in T(\bar\rho)}$ is a representation of G on a finite free $\mathcal{O}$-module W_σ. Let f denote the natural map $X_V \to X_U$, Calegari and Geraghty in Section 3.9.1 define vector bundles on X_U

$$\mathcal{L}_\sigma := (f_*(\mathcal{O}_{X_V} \otimes_{\mathcal{O}} W_\sigma))^G \ \text{ and } \ \mathcal{L}_\sigma^{\mathrm{sub}} := (f_*(\mathcal{O}_{X_V}(-\infty) \otimes_{\mathcal{O}} W_\sigma))^G ,$$

where G acts diagonally in both cases.

[7]Making the needed arrangements when these curves are stacks and not proper schemes, see the discussion in Remark 3.10 in [CG18].

Lemma 1.4.1 (Lemma 3.27 of [CG18].). *The sheaves $\mathcal{L}_\sigma$ and $\mathcal{L}_\sigma^{sub}$ defined above are locally free of finite rank on X_U.*

We will now consider the following modular forms.

Definition 1.4.2. Given an $\mathcal{O}$-module A, we call *modular forms of weight* 1 *and level* $N = N(\bar\rho)$ *with coefficients in* A elements of $\mathrm{H}^0(X_U, (\omega \otimes \mathcal{L}_\sigma)_A)$. We will denote this module by $\mathcal{M}_1(X_U, A)$

Remark 1.4.3. We remark that under Hypothesis 2, the curve X_U is just the curve $X = X_H(N)$ defined at the beginning of section 1.3.1 and the vector bundle $\mathcal{L}_\sigma$ is just the trivial sheaf $\mathcal{O}_X$. Therefore the above definition and Definition 1.3.1 of modular forms of weight 1 with coefficients in an $\mathcal{O}$-module A agree.

1.4.2 Hecke Algebras and Homology

In [CG18], Calegari and Geraghty construct Hecke operators T_ℓ for primes $\ell \notin S(\bar\rho) \cup \{p\}$ and diamond operators $\langle a \rangle$ for integers a coprime to elements of $S(\bar\rho)$ acting on the spaces of modular forms $\mathcal{M}_1(X_U, K/\mathcal{O}) = \mathrm{H}^0(X_U, (\omega \otimes \mathcal{L}_\sigma)_{K/\mathcal{O}})$.

Let $\sigma^* := \mathrm{Hom}_{\mathcal{O}}(W_\sigma, \mathcal{O})$ be the dual representation of σ and let $\mathcal{L}_\sigma^*$ be the dual bundle. Then Calegari and Geraghty[8] show that there is an injection $\mathcal{L}_{\sigma^*} \hookrightarrow \mathcal{L}_\sigma^*$ that restricts to an isomorphism

$$\mathcal{L}_{\sigma^*}^{\mathrm{sub}} \xrightarrow{\sim} \mathcal{L}_\sigma^*(-\infty) .$$

Consider now the homology $\mathrm{H}_0(X_U, \omega \otimes \mathcal{L}_\sigma)$. Using Kodaira-Spencer, one has that

$$\Omega^1_{X_U/\mathcal{O}} \otimes \mathcal{L}_\sigma^* \simeq \omega^2(-\infty) \otimes \mathcal{L}_\sigma^* \simeq \omega^2 \otimes \mathcal{L}_{\sigma^*}^{\mathrm{sub}} .$$

Now, using the same reasoning as in section 1.3.3, one has

$$\mathrm{H}_0(X_U, \omega \otimes \mathcal{L}_\sigma) = \left(\mathrm{H}^0(X_U, (\omega \otimes \mathcal{L}_{\sigma^*}^{\mathrm{sub}})_{K/\mathcal{O}})\right)^\vee .$$

Let $\mathbf{T}_\emptyset$ denote the ring of Hecke operators acting on $\mathcal{M}_1(X_U, K/\mathcal{O}) = \mathrm{H}^0(X_U, (\omega \otimes \mathcal{L}_\sigma)_{K/\mathcal{O}})$ generated by Hecke operators away from $S(\bar\rho) \cup \{p\}$; and $\mathfrak{m}_\emptyset$ the non-Eisenstein ideal generated by ϖ, $T_\ell - \mathrm{tr}(\bar\rho(\mathrm{Frob}_\ell))$ for primes outside $S(\bar\rho) \cup \{p\}$, $\langle a \rangle - \det(\bar\rho(\mathrm{Frob}_a))$ for integers a coprime to elements in $S(\bar\rho)$. Then one has an isomorphism (See proof of Theorem 3.30 in [CG18].)

$$\left(\mathrm{H}^0(X_U, (\omega \otimes \mathcal{L}_\sigma^{\mathrm{sub}})_{K/\mathcal{O}})\right)_{\mathfrak{m}_\emptyset} \xrightarrow{\sim} \left(\mathrm{H}^0(X_U, (\omega \otimes \mathcal{L}_\sigma)_{K/\mathcal{O}})\right)_{\mathfrak{m}_\emptyset} ,$$

which allows us to endow $\mathrm{H}_0(X_U, \omega \otimes \mathcal{L}_\sigma)$ with a Hecke action of $\mathbf{T}_{\emptyset, \mathfrak{m}_\emptyset}$.

In the proof of Theorem 3.30 of [CG18], Calegari and Geraghty show that there exists a minimal deformation of $\bar\rho$

$$\rho_\emptyset : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbf{T}_{\emptyset, \mathfrak{m}_\emptyset}) ,$$

which is in particular unramified at all primes $\ell \notin S(\bar\rho) \cup \{p\}$ and at these primes $\mathrm{tr}(\rho_\emptyset(\mathrm{Frob}_\ell)) = T_\ell$. From this, they deduce:

Theorem 1.4.4 (Theorem 3.30 of [CG18]). *The surjective map $\varphi : R^{min} \to \mathbf{T}_{\emptyset, \mathfrak{m}_\emptyset}$ is an isomorphism in $\mathcal{C}_{\mathcal{O}}$. Moreover $\mathbf{T}_{\emptyset, \mathfrak{m}_\emptyset}$ acts freely on $\mathrm{H}_0(X_U, \omega \otimes \mathcal{L}_\sigma)_{\mathfrak{m}_\emptyset}$. In particular, $\mathrm{H}_0(X_U, \omega \otimes \mathcal{L}_\sigma)_{\mathfrak{m}_\emptyset}$ is a $\mathbf{T}_{\emptyset, \mathfrak{m}_\emptyset}$-module of rank* 1, *when $\mathrm{H}_0(X_U, \omega_K \otimes \mathcal{L}_\sigma)_{\mathfrak{m}_\emptyset}$ is not zero.*

Following the steps of Section 1.3.5 and using the above result, one gets a proof of Theorem 1.1.1, without Hypothesis 2.

[8] Lemma 3.28 of [CG18].

Part II

Hilbert Modular Forms

Introduction

In this introduction, we briefly summarize what is done in the various Chapters of Part II of this thesis.

In Chapter 2, we recall the various models for the Hilbert moduli variety and we discuss the construction of the Shimura variety associated to the group $\mathrm{Res}_{\mathbb{Q}}^{F} \mathrm{GL}_{2,F}$. We will consider Hilbert modular forms *à la Wiles* living on an automorphic line bundle on the Shimura variety. This is done in order to have a good Hecke theory, in the sense of the attached Galois representations. We then proceed to recall how to construct the cusps of the Hilbert modular variety, the associated Tate objects and how to trivialize the sheaf of Hilbert modular forms at the cusps. We will finish this chapter by giving an explicit construction of geometric q-expansions and by showing how changing cusps changes the q-expansion.

In Chapter 3, we compute the action on geometric q expansions of the normalized $T_{\mathfrak{p}}$ operator defined by Emerton, Reduzzi and Xiao in [ERX17a]. We first recall how to properly normalize diamond operators, and we then proceed to recall the construction of $T_{\mathfrak{p}}$. Following this construction, we will be able to compute its action on geometric q-expansion on Hilbert modular forms modulo ϖ^m.

Finally, in Chapter 4, we will prove that a partial weight one Hilbert modular form, with parallel weight one for the places above a prime $\mathfrak{p}$, has associated modulo ϖ Galois representation that is unramified at $\mathfrak{p}$. In order to prove this theorem, we will use generalized partial Hasse invariants as defined by Reduzzi and Xiao in [RX17] and an exceptional sheaf adapted from their work. We will then apply the strategy of Dimitrov and Wiese in [DW18] to prove our theorem. We will make use of the computations of Chapter 3.

Notation

Let F be a totally real field of degree $d \geq 2$, with ring of integers $\mathcal{O}_F$ and different $\mathfrak{d} = \mathfrak{d}_{F/\mathbb{Q}}$. For any $x \in F$, we will denote $\mathrm{Nm}(x) := \mathrm{Nm}_{F/\mathbb{Q}}(x)$, and for any fractional ideal I of F, we will also denote by $\mathrm{Nm}(I)$ the ideal norm. Let p be a rational prime and take $\mathfrak{n}$ an integral ideal of $\mathcal{O}_F$ coprime with p. This will be the level of our Hilbert modular forms. Let $\mathcal{O}_{F,\mathfrak{n}}^\times$ denote the set of totally positive units $u \in \mathcal{O}_{F,+}^\times$ such that $u \equiv 1 \mod \mathfrak{n}$. We will need the group of units $\mathrm{E} := \mathcal{O}_{F,+}^\times/(\mathcal{O}_{F,\mathfrak{n}}^\times)^2$.

We fix $\mathfrak{C} = \{\mathfrak{c}_1, \ldots, \mathfrak{c}_{h^+}\}$ a set of representatives of the elements of the narrow class group Cl_F^+. Without loss of generality we can suppose $\mathfrak{c} \in \mathfrak{C}$ to be coprime with p. We will denote with $\mathfrak{c}_+ = \mathfrak{c} \cap F_+^\times$ the cone of totally positive elements.

Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, together with an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let Σ denote the set of embeddings $F \hookrightarrow \overline{\mathbb{Q}}$, which is also identified with the set of embeddings of F into $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$.

Let K be a finite extension of $\mathbb{Q}_p$ such that $\tau(F) \subset K$ for all $\tau \in \Sigma$. Let $\mathcal{O}$ denote its ring of integers of uniformizer ϖ and residue field $\mathbb{F} = \mathcal{O}/\varpi$. We will also identify Σ with the following sets $\{\tau : F \hookrightarrow K\}$ and $\{\tau : \mathcal{O}_F \hookrightarrow \mathcal{O}\}$. For $\mu \in F$ and $k = (k_\tau)_\tau \in \mathbb{Z}^\Sigma$, we set $\mu^k := \prod_{\tau \in \Sigma} \tau(\mu)^{k_\tau}$. The weights of our Hilbert modular forms will be elements $(k, \ell) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$. In particular, we will denote by $\mathtt{t} \in \mathbb{Z}^\Sigma$ the weight vector which has 1 in all entries.

Let $\mathfrak{p}$ a prime in $\mathcal{O}_F$ above p and let $e_\mathfrak{p}$ denote its absolute ramification index and $f_\mathfrak{p}$ its residue degree. We will denote $\Sigma_\mathfrak{p}$ the subset of Σ consisting of all p-adic embeddings of F inducing the p-adic place $\mathfrak{p}$. Let Fr denote the arithmetic Frobenius on $\overline{\mathbb{F}}_p$ and let us label the embeddings of $\mathbb{F}_\mathfrak{p} = \mathcal{O}_F/\mathfrak{p} \hookrightarrow \mathbb{F}$ as $\{\tau_{p,j} : j \in \mathbb{Z}/f_\mathfrak{p}\mathbb{Z}\}$ so that $\mathrm{Fr} \circ \tau_{\mathfrak{p},j} = \tau_{\mathfrak{p},j+1}$ for all $j \in \mathbb{Z}/f_\mathfrak{p}\mathbb{Z}$. For each $j \in \{1, \ldots, f_\mathfrak{p}\}$, there are exactly $e_\mathfrak{p}$ elements in $\Sigma_\mathfrak{p}$ that induce the embedding $W(\mathcal{O}_F/\mathfrak{p}) \to \mathcal{O}$, which we will denote $\{\tau_{\mathfrak{p},j}^{(1)}, \ldots, \tau_{\mathfrak{p},j}^{(e_\mathfrak{p})}\}$. For every $\mathfrak{p}$ in $\mathcal{O}_F$, we fix a uniformizer $\varpi_\mathfrak{p}$ for $\mathcal{O}_{F,\mathfrak{p}}$.

Chapter 2

Geometric Hilbert Modular Forms

In this chapter, we will recall the geometric construction of Hilbert modular forms. In particular, we will recall and describe the Pappas-Rapoport ([PR05]), the Deligne-Pappas ([DP94]) and the Rapoport ([Rap78]) models for the Hilbert-Blumenthal moduli space. We will then proceed to construct and compare the toroidal and minimal compactifications of the obtained Hilbert modular varieties. Finally, we will discuss the construction and the properties of the automorphic sheaves of modular forms.
We point out to the reader that we will mainly work with *à la Wiles* Hilbert modular forms, which correspond to Katz modular forms that are invariant under the action of a finite group of units of the totally real number field F. This is necessary in order to have a good Hecke theory, in the sense of attached Galois representations. These forms will be global sections of a line bundle living on a Shimura variety associated to $\mathrm{Res}^F_{\mathbb{Q}} \mathrm{GL}_{2,F}$.
Finally, we will recall in Section (2.3) how to construct the cusps of the Hilbert modular variety, the associated Tate objects and how to trivialize the sheaf of Hilbert modular forms at the cusps. We will finish this chapter by giving an explicit construction of geometric q-expansions and by showing how changing cusps changes the q-expansion.

2.1 Hilbert modular varieties and Shimura varieties

2.1.1 Hilbert-Blumenthal Abelian Schemes

Let S be a locally Noetherian $\mathcal{O}$-scheme. A *Hilbert-Blumenthal abelian scheme* (HBAS) over S is an abelian scheme $\pi : A \to S$ of relative dimension d, together with a ring embedding $\mathcal{O}_F \hookrightarrow \mathrm{End}(A/S)$ (also called *real multiplication by* $\mathcal{O}_F$). For any HBAS A/S, we have a natural direct sum decomposition

$$\pi_*\Omega^1_{A/S} = \bigoplus_{\mathfrak{p}|p} \omega_{A/S,\mathfrak{p}} = \bigoplus_{\mathfrak{p}|p} \bigoplus_{j=1}^{f_\mathfrak{p}} \omega_{A/S,\mathfrak{p},j} \ ,$$

where each $\omega_{A/S,\mathfrak{p},j}$ is locally free $\mathcal{O}_S$-module of rank $e_\mathfrak{p}$ and in particular, $W(\mathbb{F}_\mathfrak{p}) \subseteq \mathcal{O}_{F,\mathfrak{p}}$ acts on each $\omega_{A/S,\mathfrak{p},j}$ via $\tau_{\mathfrak{p},j}$. One also has a natural direct sum decomposition of the first degree de Rham cohomology

$$\mathcal{H}^1_{\mathrm{dR}}(A/S) = \bigoplus_{\mathfrak{p}|p} \bigoplus_{j=1}^{f_\mathfrak{p}} \mathcal{H}^1_{\mathrm{dR}}(A/S)_{\mathfrak{p},j} \ ,$$

where each $\mathcal{H}^1_{\mathrm{dR}}(A/S)_{\mathfrak{p},j}$ is a locally free $\mathcal{O}_S$-module of rank $2e_{\mathfrak{p}}$, since $\mathcal{H}^1_{\mathrm{dR}}(A/S)$ is locally free of rank 2 over $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S$ ([Rap78, Lemme 1.3]). Again $W(\mathbb{F}_{\mathfrak{p}}) \subseteq \mathcal{O}_{F,\mathfrak{p}}$ acts on each $\mathcal{H}^1_{\mathrm{dR}}(A/S)_{\mathfrak{p},j}$ via $\tau_{\mathfrak{p},j}$.
Let $\mathfrak{c} \in \mathfrak{C}$ be a fractional ideal of F and A a Hilbert-Blumenthal abelian scheme over S. We recall that the functor on S-schemes $(A \otimes_{\mathcal{O}_F} \mathfrak{c})$, given by $A(T) \otimes_{\mathcal{O}_F} \mathfrak{c}$, is representable by a HBAV over S. A $\mathfrak{c}$-*polarization* on a Hilbert-Blumenthal abelian scheme A/S is an S-isomorphism

$$\lambda : A \otimes_{\mathcal{O}_F} \mathfrak{c} \xrightarrow{\sim} A^{\vee} ,$$

such that the induced $\mathcal{O}_F$-linear isomorphism $\mathrm{Hom}_{\mathcal{O}_F}(A, A \otimes_{\mathcal{O}_F} \mathfrak{c}) \simeq \mathrm{Hom}_{\mathcal{O}_F}(A, A^{\vee})$ maps $\mathfrak{c}$, respectively $\mathfrak{c}_+$, onto the $\mathcal{O}_F$-module of symmetric elements $\mathrm{Sym}(A/S)$, respectively onto the cone of polarizations $\mathrm{Pol}(A/S)$.
Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_F$ coprime with p. A $\mu_{\mathfrak{n}}$-*level structure* on a Hilbert-Blumenthal abelian scheme A/S is an $\mathcal{O}_F$-linear closed immersion of group S-schemes

$$\mu : \mu_{\mathfrak{n}} \otimes \mathfrak{d}^{-1} \hookrightarrow A ,$$

where μ_n denotes the reduced sub-scheme of $\mathbb{G}_m \otimes \mathfrak{d}^{-1}$ defined as the intersection of the kernels of multiplication by elements of $\mathfrak{n}$.
Throughout the thesis, we will make the following assumption.

Hypothesis 3. Assume that $\mathfrak{n}$ does not divide $2, 3$ nor $\mathrm{Nm}(\mathfrak{d})$.

2.1.2 Models of the Hilbert Modular Variety

Historically, the first model for the Hilbert-Blumenthal moduli space Y^{R} was introduced by Rapoport ([Rap78, Definition 1.1]), where he supposed that the points of the moduli space, which are HBAS $\pi : A \to S$, are such that the cotangent space $\pi_*\Omega^1_{A/S}$ is a locally free $\mathcal{O}_F \otimes \mathcal{O}_S$-module of rank 1 (see Definition 2.1.3). In particular, Y^{R} is a smooth $\mathbb{Z}[1/\mathrm{Nm}(\mathfrak{n})]$-scheme ([Rap78, Lemme 1.23]). However, for characteristics dividing the different $\mathfrak{d}$ it is not a proper scheme (*singularité à distance finie*). This was first remarked by Deligne et Pappas, who defined a new moduli problem giving rise to a proper smooth $\mathbb{Z}[1/\mathrm{Nm}(\mathfrak{n}\mathfrak{d})]$-scheme[1] Y^{DP} ([DP94, Théorème 2.2]), which is also normal ([DP94, Corollaire 2.3]) and admits Y^{R} as an open dense subscheme. The Deligne-Pappas model is not ideal when working in characteristic $p|\mathrm{Nm}(\mathfrak{d})$, since $\mathrm{Y}^{\mathrm{DP}}_{\mathbb{F}}$ is not smooth, and for such a prime p ramifying in F there is a lack of partial Hasse invariants as defined by Andreatta and Goren ([AG05, Section 7]). Pappas and Rapoport ([PR05]) then introduced what is now known as the *splitting model* for Hilbert modular varieties, denoted here by Y^{PR}, which was later made explicit by Sasaki ([Sas19]). The advantage of this moduli space is that it allows us to work also with primes p that ramify in F. Moreover, Y^{PR} is smooth over $\mathcal{O}$ ([Sas19, Proposition 6] or [RX17, Theorem 2.9]), and Reduzzi and Xiao constructed in [RX17, Section 3] partial Hasse invariants living on $\mathrm{Y}^{\mathrm{PR}}_{\mathbb{F}}$. In what follows, we will make all of the above explicit working over $\mathrm{Spec}(\mathcal{O})$.

Let us start by defining the splitting model of the Hilbert modular variety as introduced by Pappas-Rapoport ([PR05]), as defined by Reduzzi and Xiao in ([RX17, Section 2.2]).

Definition 2.1.1. For a fractional ideal $\mathfrak{c} \in \mathfrak{C}$, let $\mathcal{M}^{\mathrm{PR}}_{\mathfrak{c}} = \mathcal{M}^{\mathrm{PR}}_{\mathfrak{c}}(\mathfrak{n})$ be the functor associating to an $\mathcal{O}$-scheme S the set of isomorphism classes of data $(A, \lambda, \mu, \mathcal{F})$, where

[1] This explains Hypothesis 3.

- (A, λ) is a $\mathfrak{c}$-polarized HBAS over S;
- μ is a $\mu_{\mathfrak{n}}$ level structure.
- $\mathcal{F}$ is a collection $(\mathcal{F}_{\mathfrak{p},j}^{(i)})_{\mathfrak{p}|p;j=1,\ldots,f_{\mathfrak{p}};i=1,\ldots,e_{\mathfrak{p}}}$ of locally free sheaves over S such that
  - $0 = \mathcal{F}_{\mathfrak{p},j}^{(0)} \subset \mathcal{F}_{\mathfrak{p},j}^{(1)} \subset \ldots \subset \mathcal{F}_{\mathfrak{p},j}^{(e_{\mathfrak{p}})} = \omega_{A/S,\mathfrak{p},j}$ and each $\mathcal{F}_{\mathfrak{p},j}^{(i)}$ is stable under $\mathcal{O}_F$-action;
  - each subquotient $\mathcal{F}_{\mathfrak{p},j}^{(i)}/\mathcal{F}_{\mathfrak{p},j}^{(i-1)}$ is a locally free $\mathcal{O}_S$-module of rank one (and hence the rank of $\mathcal{F}_{\mathfrak{p},j}^{(i)}$ is i);
  - the action of $\mathcal{O}_F$ on each subquotient $\mathcal{F}_{\mathfrak{p},j}^{(i)}/\mathcal{F}_{\mathfrak{p},j}^{(i-1)}$ factors through $\tau_{\mathfrak{p},j}^{(i)} : \mathcal{O}_F \hookrightarrow \mathcal{O}$, or equivalently, $\mathcal{F}_{\mathfrak{p},j}^{(i)}/\mathcal{F}_{\mathfrak{p},j}^{(i-1)}$ is annihilated by $[\varpi_{\mathfrak{p}}] - \tau_{\mathfrak{p},j}^{(i)}(\varpi_{\mathfrak{p}})$, where $[\varpi_{\mathfrak{p}}]$ denotes the action of $\varpi_{\mathfrak{p}}$ as an element of $\mathcal{O}_{F,\mathfrak{p}}$.

Under Hypothesis 3, this functor is representable by an $\mathcal{O}$-scheme of finite type that we will denote $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{PR}}$ ([RX17, Proposition 2.4 (1)]). Moreover, $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{PR}}$ is a smooth $\mathcal{O}$-scheme ([Sas19, Proposition 6]). We call the space

$$\mathrm{Y}^{\mathrm{PR}} := \coprod_{\mathfrak{c}\in\mathfrak{C}} \mathrm{Y}_{\mathfrak{c}}^{\mathrm{PR}}$$

the *Pappas-Rapoport moduli space*. For any $\mathcal{O}$-algebra R, we will denote $\mathrm{Y}_R^{\mathrm{PR}}$ the base change of the moduli space to R.

We point out that in general the Pappas-Rapoport functor $\mathcal{M}_{\mathfrak{c}}^{\mathrm{PR}}$ depends on the choice of ordering $\{\tau_{\mathfrak{p},j}^{(1)}, \ldots, \tau_{\mathfrak{p},j}^{(e_{\mathfrak{p}})}\}$ of the p-adic embeddings of F for every $\mathfrak{p}|p$ (see Notation). The dependence disappears when one base changes to $\mathbb{F}$, however Hilbert modular forms over $\mathbb{F}$ will still depend on this ordering, since they are defined through the integral model (see ([RX17, Remark 2.3])).

Let us now introduce the Deligne-Pappas model, which can be obtained from the Pappas-Rapoport model by forgetting the filtration.

Definition 2.1.2. For a fractional ideal $\mathfrak{c} \in \mathfrak{C}$, let $\mathcal{M}_{\mathfrak{c}}^{\mathrm{DP}} = \mathcal{M}_{\mathfrak{c}}^{\mathrm{DP}}(\mathfrak{n})$ denote the scheme representing the functor associating to an $\mathcal{O}$-scheme S the set of isomorphism classes of data (A, λ, μ), where

- (A, λ) is a $\mathfrak{c}$-polarized HBAS over S;
- μ is a $\mu_{\mathfrak{n}}$ level structure.

Again, under Hypothesis 3, this functor is representable by an $\mathcal{O}$-scheme of finite type ([RX17, Proposition 2.4 (1)]) that we will denote $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}$ and by [DP94, Corollaire 2.3] it is a normal $\mathcal{O}$-scheme. We call the space $\mathrm{Y}^{\mathrm{DP}} := \coprod_{\mathfrak{c}\in\mathfrak{C}} \mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}$ the *Deligne-Pappas moduli space*.

As remarked in the introduction of this section, the Deligne-Pappas moduli space is not smooth, but it admits an open dense subscheme which is smooth.

Definition 2.1.3. Let $\mathrm{Y}^{\mathrm{R}}_{\mathfrak{c}}$ denote the open subscheme of $\mathrm{Y}^{\mathrm{PR}}_{\mathfrak{c}}$ classifying $\mathfrak{c}$-polarized HBAS $\pi : A \to S$ satisfying the following, called *Rapoport condition*[2]

$$\pi_* \Omega^1_{A/S} \text{ is a locally free } \mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S\text{-module of rank 1.} \tag{R}$$

The open subscheme $\mathrm{Y}^{\mathrm{R}}_{\mathfrak{c}}$ is called the *Rapoport locus.* The scheme $\mathrm{Y}^{\mathrm{R}} := \coprod_{\mathfrak{c}} \mathrm{Y}^{\mathrm{R}}_{\mathfrak{c}}$ is the smooth locus of Y^{DP}.

It is clear from the definiton of $\mathrm{Y}^{\mathrm{DP}}_{\mathfrak{c}}$ and $\mathrm{Y}^{\mathrm{PR}}_{\mathfrak{c}}$ that for every $\mathfrak{c} \in \mathfrak{C}$, there is a natural forgetful map

$$\pi_{\mathfrak{c}} : \mathrm{Y}^{\mathrm{PR}}_{\mathfrak{c}} \twoheadrightarrow \mathrm{Y}^{\mathrm{DP}}_{\mathfrak{c}} \ ,$$

which is projective and it induces an isomorphism from an open subscheme of $\mathrm{Y}^{\mathrm{PR}}_{\mathfrak{c}}$ to the open subscheme $\mathrm{Y}^{\mathrm{R}}_{\mathfrak{c}}$ of $\mathrm{Y}^{\mathrm{DP}}_{\mathfrak{c}}$ ([RX17, Proposition 2.4]).

Let us now recall here some of the properties of the above defined schemes.

- For characteristics away from $\mathrm{Nm}(\mathfrak{d})$, the Deligne-Pappas and Rapoport moduli spaces coincide (see [DP94, Section 2.10]). When p ramifies in F, the Rapoport locus $\mathrm{Y}^{\mathrm{R}}_{\mathbb{F}}$ is open and dense in $\mathrm{Y}^{\mathrm{DP}}_{\mathbb{F}}$ with a complement of dimension 2 ([DP94, Théorème 2.2]).
- When p is unramified in F, the models agree over $\mathcal{O}$ (and in particular over $\mathbb{F}$) in the sense that $\mathrm{Y}^{\mathrm{PR}} = \mathrm{Y}^{\mathrm{DP}} = \mathrm{Y}^{\mathrm{R}}$ ([RX17, Introduction]).

2.1.3 Unit Actions and Shimura Variety

As already explained in the introduction, for $? \in \{\mathrm{R}, \mathrm{DP}, \mathrm{PR}\}$ the moduli spaces $\mathrm{Y}^?$ do not have a good Hecke theory and therefore one has to work with the corresponding Shimura varieties, which will be quotients of $\mathrm{Y}^?$ by a finite group of totally positive units of $\mathcal{O}_F$. Here we detail the action of $\mathcal{O}^\times_{F,+}$ and the construction of the corresponding Shimura varieties.

The functors $\mathcal{M}^{\mathrm{PR}}_{\mathfrak{c}}$ and $\mathcal{M}^{\mathrm{DP}}_{\mathfrak{c}}$ carry an action of $\mathcal{O}^\times_{F,+}$. An element $\varepsilon \in \mathcal{O}^\times_{F,+}$ acts via

$$\varepsilon : (A, \lambda, \mu, \mathcal{F}) \mapsto (A, \varepsilon\lambda, \mu, \mathcal{F}) \ . \tag{2.1}$$

In particular, this action is trivial on the subgroup $(\mathcal{O}^\times_{F,\mathfrak{n}})^2 \subset \mathcal{O}^\times_{F,+}$, where $\mathcal{O}^\times_{F,\mathfrak{n}} := \{u \in \mathcal{O}_F : u \equiv 1 \mod \mathfrak{n}\}$. In fact, for $u \in (\mathcal{O}^\times_{F,\mathfrak{n}})^2$ one has an isomorphism of $(A, \lambda, \mu, \mathcal{F}) \simeq (A, u^2\lambda, u\mu, \mathcal{F})$. Let us see why. For an abelian scheme A, multiplication by $u \in \mathcal{O}^\times_{F,\mathfrak{n}}$ defines an isomorphism of S-schemes $A \underset{\sim}{\xrightarrow{\cdot u}} A$, which induces an isomorphism on the dual abelian scheme $A^\vee \underset{\sim}{\xleftarrow{\cdot u}} A^\vee$. This isomorphism gives rise to the following commutative diagram for the $\mathfrak{c}$-polarization λ:

$$\begin{tikzcd} A \otimes_{\mathcal{O}_F} \mathfrak{c} \arrow[r, "u\otimes 1"] \arrow[d, "u^2\lambda"] & A \otimes_{\mathcal{O}_F} \mathfrak{c} \arrow[d, "\lambda"] \\ A^\vee & A^\vee \arrow[l, "\cdot u"'] \end{tikzcd}$$

[2]This originates from [Rap78, Définition 1.1].

Therefore, (A,λ) and $(A,u^2\lambda)$ belong to the same isomorphism class.
Moreover for $u \in \mathcal{O}_{F,\mathfrak{n}}^\times$, one has that the level structure $u\mu$, constructed via the commutative diagram

$$\begin{tikzcd}
\mu_{\mathfrak{n}} \otimes \mathfrak{d}^{-1} \arrow[r, hook, "\mu"] \arrow[dr, hook, "u\mu"'] & A \arrow[d, "\cdot u"] \\
 & A
\end{tikzcd}$$

is such that $u\mu = \mu$, since $u \equiv 1 \mod \mathfrak{n}$. Therefore for an element $u \in \mathcal{O}_{F,\mathfrak{n}}^\times$, one has that

$$u^2 : (A,\lambda,\mu,\mathcal{F}) \mapsto (A,u^2\lambda,\mu,\mathcal{F}) = (A,u^2\lambda,u\mu,\mathcal{F}) \simeq (A,\lambda,\mu,\mathcal{F}),$$

so u^2 acts trivially on geometric points of $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{PR}}$ and of $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}$.
In what follows, we will denote

$$\mathrm{E} := \mathcal{O}_{F,+}^\times/(\mathcal{O}_{F,\mathfrak{n}}^\times)^2 \ , \tag{2.2}$$

and we will denote by $[\varepsilon]$ the action of $\varepsilon \in \mathrm{E}$ on geometric points of $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{PR}}$ or of $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}$:

$$[\varepsilon] : (A,\lambda,\mu,\mathcal{F}) \mapsto (A,\varepsilon\lambda,\mu,\mathcal{F}) \ . \tag{2.3}$$

Proposition 2.1.4 (Reduzzi-Xiao, Proposition 2.4.(4) [RX17])**.** *For $\mathfrak{n}$ sufficiently divisible, the group* E *acts freely on the geometric points of* $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}$ *and* $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{PR}}$*. In particular, the corresponding quotients:*

$$\mathrm{Sh}_{\mathfrak{c}}^{\mathrm{PR}} = \mathrm{Y}_{\mathfrak{c}}^{\mathrm{PR}}/\mathrm{E} \quad \textit{and} \quad \mathrm{Sh}_{\mathfrak{c}}^{\mathrm{DP}} = \mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}/\mathrm{E}$$

are $\mathcal{O}$-schemes of finite type and the quotient morphisms are étale.

Form now on, we will assume the following:

Hypothesis 4. Assume Hypothesis 3 and that $\mathfrak{n}$ is sufficiently divisible[3], as in the sense of Emerton, Reduzzi and Xiao (see [ERX17b, Section 2.1.1]).

For $? \in \{\mathrm{PR},\mathrm{DP}\}$, we set

$$\mathrm{Sh}^? := \coprod_{\mathfrak{c}\in\mathfrak{C}} \mathrm{Sh}_{\mathfrak{c}}^? \ .$$

These varieties are Shimura varieties for the group $\mathrm{Res}_{\mathbb{Q}}^F \mathrm{GL}_2$, which explains the notation.

2.1.4 Compactifications

Rapoport ([Rap78, Section 5]) was the first to construct a *toroidal compactification* for Y^{R}, which over $\mathbb{C}$ reduces to a toroidal Mumford compactification. This construction was later extended to the Deligne-Pappas models and to the associated Shimura varieties by works of Dimitrov ([Dim04]). We will mainly focus in this section to recall the construction of $\mathrm{Sh}^{\mathrm{PR,tor}}$, without going in the details of how toroidal compactifications are constructed, which can be found in [Dim04].
For any $\mathfrak{c} \in \mathfrak{C}$ fix a rational polyhedral admissible cone decomposition for each isomorphism class of a cusp (see Section 2.3), which here we omit from the notation. By [Dim04, Théorème 7.2

[3]This is defined in [ERX17b, 2.1.1]. An ideal $\mathfrak{n}$ of $\mathcal{O}_F$ is said to be *sufficiently divisible* if for any CM-extensions L/F such that $\mathcal{O}_F^\times \subsetneq \mathcal{O}_L^\times$ and for any $\alpha \in \mathcal{O}_L^\times/\mathcal{O}_F^\times$, $\mathfrak{n} \subset \mathfrak{q}$ for all primes $\mathfrak{q}$ of F, inert in L and such that the image of α in $(\mathcal{O}_L/\mathfrak{q})^\times$ does not belong to $(\mathcal{O}_F/\mathfrak{q})^\times$.

(i)], there exists a smooth $\mathcal{O}$-scheme $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{R,tor}}$ containing $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{R}}$ as a fiberwise dense open subscheme, and by construction of $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{R,tor}}$ the group E acts freely on it. The toroidal compactification blows up singularities to resolve them, i.e. it replaces cusps with tori (see [Dim04, Section 2]).
For $? \in \{\mathrm{PR}, \mathrm{DP}\}$, we will denote by $\mathrm{Y}_{\mathfrak{c}}^{?,\mathrm{tor}}$ the scheme obtained by gluing $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{R,tor}}$ to $\mathrm{Y}_{\mathfrak{c}}^{?}$ over $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{R}}$. Moreover, for $? \in \{\mathrm{DP}, \mathrm{PR}\}$, we will denote

$$\mathrm{Y}^{?,\mathrm{tor}} := \coprod_{\mathfrak{c}\in\mathfrak{C}} \mathrm{Y}_{\mathfrak{c}}^{?,\mathrm{tor}} \quad \text{and} \quad \mathrm{Sh}^{?,\mathrm{tor}} := \coprod_{\mathfrak{c}\in\mathfrak{C}} \mathrm{Y}_{\mathfrak{c}}^{?,\mathrm{tor}}/\mathrm{E}\,.$$

The schemes $\mathrm{Sh}_{\mathfrak{c}}^{?,\mathrm{tor}} := \mathrm{Y}_{\mathfrak{c}}^{?,\mathrm{tor}}/\mathrm{E}$ are proper ([Dim04, Théorème 7.7]) and smooth over $\mathrm{Spec}(\mathcal{O})$ ([Dim04, Corollaire 7.5]).
The toroidal compactification is not canonical in any way, since it depends on the chosen polyhedral admissible cone decomposition. However, every choice of such a decomposition gives rise to a smooth $\mathcal{O}$-scheme.
The boundary of the toroidal compactification of Y

$$\dot{\mathrm{D}} := \mathrm{Y}^{?,\mathrm{tor}} - \mathrm{Y}^{?}$$

is a relative simple normal crossing divisor on $\mathrm{Y}^{?,\mathrm{tor}}$. The boundary divisor of the corresponding Shimura variety

$$\mathrm{D} := \mathrm{Sh}^{?,\mathrm{tor}} - \mathrm{Sh}^{?} \tag{2.4}$$

is the quotient of $\dot{\mathrm{D}}$ by the action of the group E and it is a divisor with simple normal crossings. We will use it later to define Hilbert modular cuspforms.

For every $\mathfrak{c} \in \mathfrak{C}$ and for $? \in \{\mathrm{PR}, \mathrm{DP}, \mathrm{R}\}$, let $\mathcal{A}_{\mathfrak{c}}^{?}$ denote the universal abelian scheme over $\mathrm{Y}_{\mathfrak{c}}$. Then there exists a semi-abelian scheme $\mathcal{A}_{\mathfrak{c}}^{?,\mathrm{tor}} \to \mathrm{Y}_{\mathfrak{c}}^{?,\mathrm{tor}}$ extending the universal abelian scheme $\mathcal{A}_{\mathfrak{c}}^{?} \to \mathrm{Y}_{\mathfrak{c}}^{?}$ ([Dim04, Théorème 7.2]). Set

$$\mathcal{A}^{?,\mathrm{tor}} := \coprod_{\mathfrak{c}\in\mathfrak{C}} \mathcal{A}_{\mathfrak{c}}^{?,\mathrm{tor}}\,,$$

which is the universal object over $\mathrm{Y}^{?,\mathrm{tor}}$, but it might not descend to $\mathrm{Sh}^{?}$.

Following Chai ([Cha90, Section 4]) and Dimitrov ([Dim04, Théorème 8.6]), one defines the *minimal compactification*[4] of $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}$ by

$$\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP,min}} := \mathrm{Proj}\left(\bigoplus_{k\geq 0} \mathrm{H}^0\left(\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}, \left(\wedge^d_{\mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}}} \pi_*\Omega_{\mathcal{A}_{\mathfrak{c}}^{\mathrm{DP}}/\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}}\right)^{\otimes k}\right)\right)\,,$$

where $\pi : \mathcal{A}_{\mathfrak{c}}^{\mathrm{DP}} \to \mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}$ denotes the universal abelian scheme over $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}$. The scheme $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP,min}}$ is projective, normal and of finite type ([Dim04, Théorème 8.6.(iii)]). Moreover, by [Dim04, Théorème 8.6. (ii)] for any smooth toroidal compactification $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP,tor}}$ there is a canonical projection

$$\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP,tor}} \twoheadrightarrow \mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP,min}}\,.$$

The minimal compactification $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP,min}}$ is not smooth and the boundary of $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP,min}}$ is a union of points, which has codimension d. In particular, toroidal compactifications can be seen as explicit

[4]This is also known as the Bailly-Borel-Satake compactification.

desingularizations of the minimal compactification at its cusps. By construction (see [FC90, Chapter V.2]), one has that for any smooth toroidal compactification

$$\begin{tikzcd} \mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}} \arrow[r, hook] \arrow[rr, hook, bend right, "\mathrm{Proj}"'] & \mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP,tor}} \arrow[r, two heads] & \mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP,min}} \end{tikzcd}.$$

This does not translate to the Pappas-Rapoport model, i.e. the minimal compactification of Y^{PR} cannot be constructed via the Proj. This is because the semi-abelian varieties over points of $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}$ are actual abelian varieties, and $\wedge^d_{\mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}}} \pi_* \Omega_{\mathcal{A}_{\mathfrak{c}}^{\mathrm{DP}}/\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}}$ is trivial when restricted to the boundary of $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP,tor}}$ ([Cha90, 4.4.3]). On the Pappas-Rapoport model the singularities at finite distance do contract, and $\wedge^d_{\mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP}}}} \pi_* \Omega_{\mathcal{A}_{\mathfrak{c}}^{\mathrm{PR}}/\mathrm{Y}_{\mathfrak{c}}^{\mathrm{PR}}}$ is not generated by its global sections. One then defines the minimal compactification of the Pappas-Rapoport model, denoted $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{PR,min}}$ by gluing $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{PR}}$ with $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP,min}}$ over the Rapoport locus $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{R}}$.

The action of the group E extends to an action on $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP,min}}$, and therefore the minimal compactification $\mathrm{Sh}_{\mathfrak{c}}^{\mathrm{DP,min}}$ of $\mathrm{Sh}_{\mathfrak{c}}^{\mathrm{DP}}$ is defined as the quotient $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{DP,min}}/\mathrm{E}$ (see [Dim04, Théorème 8.6 (iii)]). Denote $\mathrm{Sh}_{\mathfrak{c}}^{\mathrm{PR,min}} := \mathrm{Y}_{\mathfrak{c}}^{\mathrm{PR,min}}/\mathrm{E}$ the minimal compactification of the Shimura variety of the Pappas-Rapoport model. Again as before, for $? \in \{\mathrm{PR}, \mathrm{DP}\}$ one sets

$$\mathrm{Y}^{?,\mathrm{min}} := \coprod_{\mathfrak{c} \in \mathfrak{C}} \mathrm{Y}_{\mathfrak{c}}^{?,\mathrm{min}} \quad \text{and} \quad \mathrm{Sh}^{?,\mathrm{min}} := \coprod_{\mathfrak{c} \in \mathfrak{C}} \mathrm{Sh}_{\mathfrak{c}}^{?,\mathrm{min}} .$$

The minimal compactification does not admit a universal object over $\mathrm{Y}_{\mathfrak{c}}^{?}$ for $d > 1$ (see [Cha90]). However, this compactification is necessary to detect ampleness of sheaves, and we will use it later in Section 4.1

We conclude this section with a commutative diagram connecting most of the objects defined up until now. Let R be any $\mathcal{O}$-algebra and recall that the subscript R denotes base change. One has the following commutative diagram of $\mathcal{O}$-schemes:

$$\begin{tikzcd}
\mathrm{Y}_R^{\mathrm{PR,tor}} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{Y}_R^{\mathrm{DP,tor}} \arrow[dr] \arrow[dd] & \\
& \mathrm{Sh}_R^{\mathrm{PR,tor}} \arrow[rr] \arrow[dd] & & \mathrm{Sh}_R^{\mathrm{DP,tor}} \arrow[dd] \\
\mathrm{Y}_R^{\mathrm{PR,min}} \arrow[rr] \arrow[dr] & & \mathrm{Y}_R^{\mathrm{DP,min}} \arrow[dr] & \\
& \mathrm{Sh}_R^{\mathrm{PR,min}} \arrow[rr] & & \mathrm{Sh}_R^{\mathrm{DP,min}}
\end{tikzcd}$$

where the horizontal lines are given by forgetful maps, the vertical maps are projections, and the diagonal maps are the quotient maps with Galois group E.

2.2 Automorphic Sheaves and Geometric Hilbert Modular Forms

We will from now on focus on the Pappas-Rapoport model and we now proceed to define the sheaf of Hilbert modular forms over $\mathrm{Y} = \mathrm{Y}^{\mathrm{PR}}$ and give conditions for its existence, via descent, over $\mathrm{Sh} = \mathrm{Sh}^{\mathrm{PR}}$.

2.2.1 Automorphic line bundles over Y^{PR}

Recall that we denote by $\mathcal{A}_{\mathfrak{c}} \to \mathrm{Y}_{\mathfrak{c}}$ the universal abelian scheme over $\mathrm{Y}_{\mathfrak{c}}$, and by $\pi : \mathcal{A} = \coprod_{\mathfrak{c}} \mathcal{A}_{\mathfrak{c}} \to \mathrm{Y}$ the universal abelian scheme over Y. Let e be the zero-section of $\pi : \mathcal{A} \to \mathrm{Y}$, then

$$\omega_{\mathcal{A}/\mathrm{Y}} := e^*\Omega^1_{\mathcal{A}/\mathrm{Y}} \simeq \pi_*\Omega^1_{\mathcal{A}/\mathrm{Y}} \,.$$

Denote by $\mathcal{F} = (\mathcal{F}^{(i)}_{\mathfrak{p},j})_{\mathfrak{p}|p;j=1,\dots,f_{\mathfrak{p}};i=1,\dots,e_{\mathfrak{p}}}$ the universal filtration of $\omega_{\mathcal{A}/\mathrm{Y}}$. For each p-adic embedding $\tau = \tau^{(i)}_{\mathfrak{p},j}$ of F into $\bar{\mathbb{Q}}_p$, following [RX17, Section 2.2], we set

$$\dot{\omega}_\tau := \mathcal{F}^{(i)}_{\mathfrak{p},j}/\mathcal{F}^{(i-1)}_{\mathfrak{p},j} \,, \tag{2.5}$$

which is an automorphic[5] line bundle on Y. As explained in [RX17, Notation 2.6] each $\dot{\omega}_\tau$ does not descend in general to the Deligne-Pappas model Y^{DP}. This is because the Deligne-Pappas model does not see the filtration. However, since $\otimes_{\tau\in\Sigma}\, \dot{\omega}_\tau = \wedge^d \omega_{\mathcal{A}/\mathrm{Y}}$ is the Hodge bundle, it does descend to Y^{DP}. Following [RX17] and [ERX17a], the dot notation will be reserved for sheaves over the moduli space Y, while the notation without a dot will later denote sheaves on the Shimura variety Sh.
For a p-adic embedding τ of F and for each $\mathfrak{c} \in \mathfrak{C}$, we set

$$\dot{\delta}_\tau := \big(\wedge^2_{\mathcal{O}_F\otimes\mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}}} \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}_{\mathfrak{c}}/\mathrm{Y}_{\mathfrak{c}})\big) \otimes_{\mathcal{O}_F\otimes\mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}},\tau\otimes 1} \mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}} \,,$$

which is a trivial line bundle over $\mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}}$, since by [RX17, Lemma 2.5], one has the following canonical isomorphism

$$\dot{\delta}_\tau \simeq \big(\mathfrak{c}\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}}\big) \otimes_{\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}},\tau\otimes 1} \mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}} \,. \tag{2.6}$$

One extends the line bundle $\dot{\delta}_\tau$ to a trivial line bundle on Y, still denoted by $\dot{\delta}_\tau$. In particular, for $\tau \in \Sigma$, $\dot{\omega}_\tau \otimes_{\mathcal{O}_{\mathrm{Y}}} \dot{\delta}_\tau \simeq \dot{\omega}_\tau$.
By [RX17, Theorem 2.9], the sheaf of relative differentials $\Omega^1_{\mathrm{Y}^{\mathrm{PR}}/\mathcal{O}}$ admits a canonical *Kodaira-Spencer filtration* whose successive subquotients are given by

$$\dot{\omega}_\tau^{\otimes 2} \otimes_{\mathcal{O}_{\mathrm{Y}^{\mathrm{PR}}}} \dot{\delta}_\tau^{\otimes(-1)} \text{ for } \tau \in \Sigma. \tag{2.7}$$

We will now proceed to recall, following [RX17, Section 2.11], how to construct line bundles $\dot{\omega}^{\mathrm{tor}}, \dot{\delta}^{\mathrm{tor}}$ on the toroidal compatification $\mathrm{Y}^{\mathrm{tor}}$ that agree with the above defined ones when restricted to Y. Let us point out that, when considering Hilbert modular forms, by the Köcher principle the forms will be the same whether they are defined over the toroidal compactification or on the non-compactified moduli space.

[5] The adjective automorphic here refers to the fact that global sections of this line bundle are automorphic forms.

Let $\mathcal{A}^{\mathrm{R,tor}} \to \mathrm{Y}^{\mathrm{R,tor}}$ denote the semi-abelian scheme extending the universal abelian scheme $\mathcal{A}^{\mathrm{R}} \to \mathrm{Y}^{\mathrm{R}}$, and let e denote its unit section. The sheaf

$$\dot{\omega}^{\mathrm{R,tor}} := e^*\Omega_{\mathcal{A}^{\mathrm{R,tor}}/\mathrm{Y}^{\mathrm{R,tor}}}$$

is a locally free of rank one $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathrm{Y}^{\mathrm{R,tor}}}$-module over $\mathrm{Y}^{\mathrm{R,tor}}$. For $\tau \in \Sigma$, we set

$$\dot{\omega}_\tau^{\mathrm{R,tor}} := \dot{\omega}^{\mathrm{R,tor}} \otimes_{\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}^{\mathrm{R,tor}}},\tau\otimes 1} \mathcal{O}_{\mathrm{Y}^{\mathrm{R,tor}}} .$$

In particular, one has that $\dot{\omega}_\tau^{\mathrm{R,tor}}$ and $\dot{\omega}_\tau$ (defined in Equation 2.5) agree when they are both restricted to the open subscheme Y^{R}. Now, gluing $\dot{\omega}_\tau$ with $\dot{\omega}_\tau^{\mathrm{R,tor}}$ over the Rapoport locus Y^{R} gives a line bundle denoted $\dot{\omega}_\tau^{\mathrm{tor}}$ over $\mathrm{Y}^{\mathrm{tor}}$. In the same fashion, one extends the trivial line bundle $\dot{\delta}_\tau$ on $\mathrm{Y}_{\mathfrak{c}}$ to a (trivial) line bundle

$$\dot{\delta}_\tau^{\mathrm{tor}} \simeq \left(\mathfrak{c}\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}^{\mathrm{tor}}}\right) \otimes_{\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}^{\mathrm{tor}}},\tau\otimes 1} \mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}^{\mathrm{tor}}} .$$

We will drop the notation tor from the bundles, when it is obvious to which bundles we are refering.

As explained by Emerton, Reduzzi and Xiao in [ERX17a, Section 2.8], using results of Tian and Xiao (see [TX16, Section 2.11(4)]), one deduces from the Kodaira-Spencer filtration (Equation 2.7) canonical isomorphisms:

$$\dot{KS} : \wedge^d_{\mathcal{O}_{\mathrm{Y}^{\mathrm{tor}}}} \Omega^1_{\mathrm{Y}^{\mathrm{tor}}/\mathcal{O}}(\dot{\mathrm{D}}) \simeq \bigotimes_{\tau\in\Sigma} \left(\dot{\omega}_\tau^{\otimes 2} \otimes_{\mathcal{O}_{\mathrm{Y}^{\mathrm{tor}}}} \dot{\delta}_\tau^{\otimes(-1)}\right) \tag{2.8}$$

and

$$KS : \wedge^d_{\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}} \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}/\mathcal{O}}(\mathrm{D}) \simeq \bigotimes_{\tau\in\Sigma} \left(\omega_\tau^{\otimes 2} \otimes_{\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}} \delta_\tau^{\otimes(-1)}\right) \tag{2.9}$$

2.2.2 Unit Action and line bundles over Sh

Let us now proceed to see how the action defined in Equation (2.3) translates on sheaves and we will provide the sheaves $\dot{\omega}_\tau, \dot{\delta}_\tau$ with an action of $\mathrm{E} := \mathcal{O}_{F,+}/(\mathcal{O}_{F,\mathfrak{n}}^\times)^2$.
Following Dimitrov and Tilouine (see [DT04, Section 4]), one provides the sheaves $\dot{\omega}_\tau$ with an action of $\mathcal{O}_{F,+}^\times$: a positive unit $\varepsilon \in \mathcal{O}_{F,+}^\times$ maps a local section s of $\dot{\omega}_\tau$ to $\tau(\varepsilon)^{-1/2}[\varepsilon]^*s$ (see Equation 2.3 for the definition of $[\varepsilon]$). Let us explain why this action is trivial for the subgroup $(\mathcal{O}_{F,\mathfrak{n}}^\times)^2$. Let $u \in \mathcal{O}_{F,\mathfrak{n}}^\times$. We know that u^2 acts trivially on points of Y, because $(A, \lambda, \mu, \mathcal{F})$ has the same isomorphism class as $(A, u^2\lambda, u\mu, \mathcal{F})$. However, we recall that on the actual HBAS A the action maps A to uA, which is isomorphic to A. Therefore, on open sets $U = \operatorname{Spec} R$, this action is given by the isomorphism of $\mathcal{O}$-modules $R \xrightarrow{\cdot\tau(u)} R$, which induces an isomorphism of sheaves $(u^2)^*\dot{\omega}_\tau(U) = (\dot{\omega}_\tau(U) \otimes_R (\tau(u)R)) = \tau(u)\dot{\omega}_\tau(U)$. So $\tau(u)^{-1}(u^2)^*s = \tau^{-1}(u)\tau(u)s = s$ for any local section s of $\dot{\omega}_\tau$. Therefore the action we provided $\dot{\omega}_\tau$ with factors through the group E. Moreover, this action is well defined over K, which we can suppose to contain, via the embeddings $\tau \in \Sigma$, the field extension $F(\sqrt{\varepsilon},\ \varepsilon \in \mathcal{O}_{F,+}^\times)$.
We provide also the invertible sheaf $\dot{\delta}_\tau$ with a non-trivial action of E: an element $\varepsilon \in \mathrm{E}$ maps a local section s of $\dot{\delta}_\tau$ to $\tau(\varepsilon)^{-1}[\varepsilon]^*s$. In particular, the invertible sheaves $\dot{\omega}_\tau$ and $\dot{\delta}_\tau$ descend to invertible sheaves denoted respectively ω_τ and δ_τ on Sh, by Lemma B.1.1 and properties of the descent (see Appendix B for more details).

Lemma 2.2.1. *Let R be any $\mathcal{O}$-algebra. Then the descent of*

$$\dot{\omega}_R^{k,\ell} := \bigotimes_{\tau\in\Sigma} \left(\dot{\omega}_{\tau,R}^{\otimes k_\tau} \otimes_{\mathcal{O}_{\mathrm{Y}_R}} \dot{\delta}_{\tau,R}^{\otimes \ell_\tau}\right),$$

over Y_R to Sh_R has non-zero global sections only if $u^{k+2\ell} = \prod_{\tau\in\Sigma}\tau(u)^{k_\tau+2\ell_\tau}$ is 1 in R for all $u \in \mathcal{O}_{F,\mathfrak{n}}^\times$.

Proof. Let $\omega_R^{k,\ell}$ denote the sheaf $\bigotimes_{\tau\in\Sigma}\left(\omega_{\tau,R}^{\otimes k_\tau} \otimes_{\mathcal{O}_{\mathrm{Sh}_R}} \delta_{\tau,R}^{\otimes \ell_\tau}\right)$ over Sh_R. By properties of the descent of sheaves through a finite étale map (see Equation B.1), one can identify

$$\mathrm{H}^0(\mathrm{Sh}_R, \omega_R^{k,\ell}) = \mathrm{H}^0(\mathrm{Y}_R, \dot{\omega}_R^{k,\ell})^{\mathrm{E}},$$

which are invariant global sections under the E-action. Therefore, if an element u^2, for $u \in \mathcal{O}_{F,\mathfrak{n}}^\times$, does not act trivially on $\dot{\omega}_R^{k,\ell}$ the R-module of global sections $\mathrm{H}^0(\mathrm{Sh}_R, \omega_R^{k,\ell})$ will only contain the zero element. In particular, a section s of the sheaf $\dot{\omega}_R^{k,\ell}$ is mapped by the action of u^2 to $\prod_{\tau\in\Sigma}\tau(u)^{k_\tau+2\ell_\tau}s$. The Lemma follows. □

Remark 2.2.2. If R is a ring of characteristic 0, the above lemma is verified if and only if $k_\tau + 2\ell_\tau = \mathsf{w} \in \mathbb{Z}$ an integer independent of the embeddings. Such weights are called *paritious*, see Definition 2.2.3.
In characteristic p, one has more freedom on the weights, as long as the condition of Lemma 2.2.1 is satisfied. Forms of non-paritious weights do exists, a concrete example are the generalized partial Hasse invariants constructed by Reduzzi and Xiao in [RX17, Section 3].

For any $\tau \in \Sigma$, the line bundles $\dot{\omega}_\tau^{\mathrm{tor}}, \dot{\delta}_\tau^{\mathrm{tor}}$ constructed in the previous section descend to line bundles over the Shimura variety $\mathrm{Sh}^{\mathrm{tor}}$, where we will denote them respectively ω_τ and δ_τ. In particular, the line bundle δ_τ may not be trivial over $\mathrm{Sh}^{\mathrm{tor}}$, whereas $\otimes_{\tau\in\Sigma}\delta_\tau$ is, since $\mathcal{O}_{F,+}^\times$ acts on it as the naïve pullback (see discussion before [ERX17a, Remark 2.6]).

2.2.3 Geometric Hilbert Modular Forms

For $k, \ell \in \mathbb{Z}^\Sigma$, we define a line bundle over $\mathrm{Y}^{\mathrm{tor}}$

$$\dot{\omega}^{k,\ell} := \bigotimes_{\tau\in\Sigma} \left(\dot{\omega}_\tau^{\otimes k_\tau} \otimes_{\mathcal{O}_{\mathrm{Y}^{\mathrm{tor}}}} \dot{\delta}_\tau^{\otimes \ell_\tau}\right),$$

where by the definition of the action of E an element $u \in \mathcal{O}_{F,\mathfrak{n}}^\times$ acts via multiplication by $u^{k+2\ell} := \prod_{\tau\in\Sigma}\tau(u)^{k_\tau+2\ell_\tau}$. Moreover, for $k, \ell \in \mathbb{Z}^\Sigma$, we set

$$\omega^{k,\ell} := \bigotimes_{\tau\in\Sigma} \left(\omega_\tau^{\otimes k_\tau} \otimes_{\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}} \delta_\tau^{\otimes \ell_\tau}\right).$$

As explained in Lemma 2.2.1, in order to possibly have global section of the descended sheaf, one has to carefully choose the weights (k, ℓ) according to the base one is working with. For any $\mathcal{O}$-algebra R, we will suppose the following.

Hypothesis 5. Let R any $\mathcal{O}$-algebra. We assume that $k, \ell \in \mathbb{Z}^\Sigma$ are such that $u^{k+2\ell}$ is 1 in R, for all $u \in \mathcal{O}_{F,\mathfrak{n}}^\times$.

Under Hypothesis 5 and by Lemma 2.2.1, the line bundle $\omega_R^{k,\ell}$ is an invertible line bundle on Sh_R and it might contain non-zero Hilbert modular forms.

We will be interested in working over $\mathcal{O}$, and therefore we make the following definition.

Definition 2.2.3. Given $k, \ell \in \mathbb{Z}^\Sigma$, we say that the weight (k, ℓ) is *paritious* if $k_\tau + 2\ell_\tau = \mathsf{w}$ for all $\tau \in \Sigma$, where $\mathsf{w} \in \mathbb{Z}$ is an integer independent of τ.

In particular, when working over $\mathcal{O}$ we will be obliged to work with paritious weights.

We now have all the ingredients to define geometric Hilbert modular forms. Recall that $\mathrm{D} := \mathrm{Sh}^{\mathrm{tor}} - \mathrm{Sh}$, which is a divisor with simple normal crossing on $\mathrm{Sh}^{\mathrm{tor}}$.

Definition 2.2.4. Let $(k, \ell) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$ be a paritious weight. A *geometric Hilbert modular form of paritious weight* $\mathsf{w} = k+2\ell$ *and level* $\mathfrak{n}$ *with coefficients over* $\mathcal{O}$ is an element of $\mathrm{H}^0(\mathrm{Sh}^{\mathrm{tor}}, \omega^{k,\ell})$. We will denote this module by $\mathcal{M}_{k,\mathsf{w}}(\mathfrak{n}; \mathcal{O})$.
A *cuspidal Hilbert modular form of paritious weight* $\mathsf{w} = k + 2\ell$ *and level* $\mathfrak{n}$ *with coefficients over* $\mathcal{O}$ is an element of $\mathrm{H}^0(\mathrm{Sh}^{\mathrm{tor}}, \omega^{k,\ell}(-\mathrm{D}))$. We denote the submodule of cuspidal Hilbert modular forms by $\mathcal{S}_{k,\mathsf{w}}(\mathfrak{n}; \mathcal{O})$.

More in general, we define Hilbert modular forms of arbitrary weight on an $\mathcal{O}$-algebra R satisfying Hypothesis 5.

Definition 2.2.5. For a weight $(k, \ell) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$ and for any $\mathcal{O}$-algebra R satisfying Hypothesis 5, a *geometric Hilbert modular form of weight* (k, ℓ) *and level* $\mathfrak{n}$ *with coefficients over* R is an element of $\mathrm{H}^0(\mathrm{Sh}_R^{\mathrm{tor}}, \omega_R^{k,\ell})$. We will denote this module by $\mathcal{M}_{k,\ell}(\mathfrak{n}; R)$. A *cuspidal Hilbert modular form* is an element of the sudmodule $\mathrm{H}^0(\mathrm{Sh}_R^{\mathrm{tor}}, \omega_R^{k,\ell}(-\mathrm{D}))$, which we will denote $\mathcal{S}_{k,\ell}(\mathfrak{n}; R)$.

By definition of the Shimura variety $\mathrm{Sh}_R^{\mathrm{tor}}$, it is clear that $\mathcal{M}_{k,\ell}(\mathfrak{n}; R)$ is a direct sum as an R-module of $\mathrm{H}^0(\mathrm{Sh}_{\mathfrak{c},R}^{\mathrm{tor}}, \omega_R^{k,\ell})$, whose elements are called $\mathfrak{c}$*-polarized Hilbert modular forms*, over the fixed set of representatives $\mathfrak{C}$.
To give a better understanding of these elements, one can use Katz's description of $\mathfrak{c}$-polarized Hilbert modular forms ([Kat78, 1.2]), which we here recall as given by Reduzzi and Xiao in [RX17, Section 2.12].

Let R be an $\mathcal{O}$-algebra and let $k, \ell \in \mathbb{Z}^\Sigma$ satisfying Hypothesis 5. Let R' be an R-algebra and let $\mathfrak{c} \in \mathfrak{C}$. A $\mathfrak{c}$*-polarized test object over* R' is a tuple $(A, \lambda, \mu, \mathcal{F}, \underline{s}, \underline{t})$, where $(A, \lambda, \mu, \mathcal{F})$ is a $\mathfrak{c}$-polarized HBAS with a level $\mathfrak{n}$ structure μ and filtration $\mathcal{F}$ as described above; $\underline{s} = (s_\tau)_{\tau\in\Sigma}$ is a choice of generators for each free rank one R'-module $\dot{\omega}_{A/R',\tau}$ and analogously $\underline{t} = (t_\tau)_{\tau\in\Sigma}$ is a choice of generators for each free rank one R'-module $\dot{\delta}_{A/R',\tau}$.

Definition 2.2.6. A $\mathfrak{c}$*-polarized Katz Hilbert modular form over* R *of level* $\mathfrak{n}$ *and weights* (k, ℓ) is a rule f which assigns to any Noetherian R-algebra R' and to any $\mathfrak{c}$-polarized test object $(A, \lambda, \mu, \mathcal{F}, \underline{s}, \underline{t})$ over R' an element $f(A, \lambda, \mu, \mathcal{F}, \underline{s}, \underline{t}) \in R'$ such that

(i) $f(A, \lambda, \mu, \mathcal{F}, \underline{s}, \underline{t})$ depends only on the isomorphism class of $(A, \lambda, \mu, \mathcal{F}, \underline{s}, \underline{t})$;

(ii) it is compatible with base change in R';

(iii) it satisfies $f(A, \varepsilon\lambda, \mu, \mathcal{F}, \underline{s}, \underline{t}) = f(A, \lambda, \mu, \mathcal{F}, \underline{s}, \underline{t})$ for any $\varepsilon \in \mathcal{O}_{F,+}^\times$,

(iv) it satisfies

$$f(A, \lambda, \mu, \mathcal{F}, \underline{\alpha}\,\underline{s}, \underline{\beta}\,\underline{t}) = \left(\prod_{\tau\in\Sigma} \alpha_\tau^{-k_\tau} \beta_\tau^{-\ell_\tau}\right) f(A, \lambda, \mu, \mathcal{F}, \underline{s}, \underline{t})$$

for all $\underline{\alpha} = (\alpha_\tau)_{\tau\in\Sigma}$ and $\underline{\beta} = (\beta_\tau)_{\tau\in\Sigma}$ in $(R'^\times)^\Sigma$, where $\underline{\alpha}\,\underline{s} = (\alpha_\tau s_\tau)_{\tau\in\Sigma}$ and $\underline{\beta}\,\underline{t} = (\beta_\tau t_\tau)_{\tau\in\Sigma}$.

Remark 2.2.7. Forms not satisfying condition (iii) are elements of $\mathrm{H}^0(\mathrm{Y}_{\mathfrak{c},R}, \omega^{k,\ell}_{\mathfrak{c},R})$.

2.3 Cusps and Tate Varieties

In this section, we will recall the definition of cusps for the Hilbert modular variety Y, which are used in the construction of the toroidal compactification of this variety. We will mainly be following work of Dimitrov ([Dim04]). For a fractional ideal $\mathfrak{a}$ of F, we will denote $\mathfrak{a}^* = (\mathfrak{a}\mathfrak{d})^{-1}$.

Definition 2.3.1 (Dimitrov, Définition 3.2 [Dim04])**.** Let $\mathfrak{c} \in \mathrm{Cl}_F^+$, a $\mathfrak{c}$-*cusp of level* $\mathfrak{n}$ is an equivalence class of tuples $\mathcal{C} = (\mathfrak{a}, \mathfrak{b}, H, i, j, \Lambda, \gamma)$ where:

(i) $\mathfrak{a}, \mathfrak{b}$ are fractional ideals of F coprime with p such that $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$;

(ii) H is an $\mathcal{O}_F$-lattice of F^2 that sits in the exact sequence of $\mathcal{O}_F$-modules $0 \to \mathfrak{a}^* \xrightarrow{i} H \xrightarrow{j} \mathfrak{b} \to 0$;

(iii) $\Lambda : \wedge^2_{\mathcal{O}_F} H \xrightarrow{\sim} \mathfrak{c}^*$ is an isomorphism of $\mathcal{O}_F$-modules;

(iv) $\gamma : \mathfrak{n}^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1} \hookrightarrow \mathfrak{n}^{-1}H/H$ is an injective morphism of $\mathcal{O}_F$-modules.

for the following equivalence relation: $(\mathfrak{a}, \mathfrak{b}, H, i, j, \Lambda, \gamma)$ and $(\mathfrak{a}', \mathfrak{b}', H', i', j', \Lambda', \gamma')$ are equivalent if all the following are verified:

1. $\mathfrak{a} = \mathfrak{a}'$ and $\mathfrak{b} = \mathfrak{b}'$;
2. there exists a commutative diagram of $\mathcal{O}_F$-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a}^* \arrow[r, "i"] \arrow[d] & H \arrow[r, "j"] \arrow[d] & \mathfrak{b} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (\mathfrak{a}')^* \arrow[r, "i"] & H' \arrow[r, "j"] & \mathfrak{b}' \arrow[r] & 0 \end{tikzcd}$$

   where the vertical maps are isomorphism;
3. the isomorphism $\wedge^2_{\mathcal{O}_F} H \simeq \wedge^2_{\mathcal{O}_F} H'$ induces, via Λ and Λ', an automorphism of $\mathfrak{c}^*$ given by an element of $\mathcal{O}_{F,+}^\times$;
4. the reduction modulo $\mathfrak{n}$ of the isomorphism $H \simeq H'$ makes the following diagram commutative

$$\begin{tikzcd} \mathfrak{n}^{-1}H/H \arrow[rr, "\sim"] & & \mathfrak{n}^{-1}H'/H' \\ & \mathfrak{n}^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1} . \arrow[ul, hook, "\gamma"] \arrow[ur, hook, "\gamma'"'] & \end{tikzcd}$$

Moreover, we associate to $\mathcal{C}$ the fractional ideal $\mathfrak{b}' \supset \mathfrak{b}$ such that $\mathfrak{b}'/\mathfrak{b} = j(\mathrm{im}(\gamma))$ and the fractional ideal $X = \mathfrak{c}\mathfrak{b}\mathfrak{b}'$. The cusp is said to be *unramified* if $\mathfrak{b}' = \mathfrak{b}$.

Remark 2.3.2. The lattice H is non-canonically isomorphic to $\mathfrak{b} \oplus \mathfrak{a}^*$. By definition $X \supset \mathfrak{a}\mathfrak{b}$. For unramified cusps, $X = \mathfrak{a}\mathfrak{b}$.

Let $\mathcal{C}$ be a $\mathfrak{c}$-cusp, with associated $\mathfrak{a}, \mathfrak{b}, X$ and consider $S := \mathrm{Spec}(\mathcal{O}[[q^\xi; \xi \in X_+]])$. One fixes a smooth rational polyhedral admissible cone decomposition of X_+^* giving rise by the construction of [Dim04, Section 2] to a Tate object $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}$ defined over a suitable scheme S_X (This is a suitable subring of $\mathcal{O}[[q^\xi; \xi \in X_+]]$, which is denoted by $\bar{S}_\sigma$ in [Dim04], where $\sigma \in \Sigma^{\mathcal{C}}$ is an element of the smooth rational polyhedral admissible cone decomposition of X_+^*). Moreover, the Tate object $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}$ is a $\mathfrak{c}$-polarized abelian variety with $\mathfrak{n}$-level structure. In particular, one has the following short exact sequence of S_X-schemes,

$$0 \longrightarrow \mathfrak{b} \xrightarrow{q} \mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{a}^* \longrightarrow \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}} \longrightarrow 0$$

This Tate object comes with additional structure (polarization, level structure, basis for the differential sheaf, see [Dim04, Proof of Théorème 7.2]), and is defined over any $S_{X'}$ for X' a fractional ideal of $\mathcal{O}_F$ such that $X' \supset \mathfrak{a}\mathfrak{b}$.

Proposition 2.3.3. *Let* $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}$ *be a Tate object over the scheme* S_X*. Then there are canonical isomorphisms as* $\mathcal{O}_{S_X}$*-modules*

$$\pi_*\Omega^1_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}/S_X} \simeq \mathfrak{a} \otimes_{\mathbb{Z}} \mathcal{O}_{S_X} \tag{2.10}$$

$$\wedge^2_{\mathcal{O}_F\otimes\mathcal{O}_{Y_\mathfrak{c}}} \mathcal{H}^1_{dR}(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}/S_X) \simeq \mathfrak{c}\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathcal{O}_{S_X} \ . \tag{2.11}$$

Proof. For (2.10), see [Dim04, Équation (5)].
For (2.11), let A be an abelian scheme over a scheme S and consider the following short exact sequence ([Rap78, See discussion after Lemme 1.3])

$$0 \to \mathrm{Lie}(A/S)^\vee \to \mathcal{H}^1_{\mathrm{dR}}(A/S) \to \mathrm{Lie}(A^\vee/S) \to 0 \ .$$

Recall that $\mathrm{Lie}(A/S)$ is the tangent space at 1 of the abelian scheme A/S, while $\omega_{A/S}$ is the cotangent space at 1 for the abelian scheme A/S. Then we can reinterpret this short exact sequence as

$$0 \to \omega_{A/S} \to \mathcal{H}^1_{\mathrm{dR}}(A/S) \to \omega^\vee_{A^\vee/S} \to 0 \ .$$

Taking $A = \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}$ over $S = S_X$, and knowing that $\mathrm{Tate}^\vee_{\mathfrak{a},\mathfrak{b}} \simeq \mathrm{Tate}_{\mathfrak{b},\mathfrak{a}}$, one gets

$$0 \to \omega_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}/S_X} \to \mathcal{H}^1_{\mathrm{dR}}(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}/S_X) \to \omega^\vee_{\mathrm{Tate}_{\mathfrak{b},\mathfrak{a}}/S_X} \to 0 \ . \tag{2.12}$$

Now by definition $\omega^\vee_{\mathrm{Tate}_{\mathfrak{b},\mathfrak{a}}/S_X} = \mathrm{Hom}_{\mathcal{O}_F}(\omega_{\mathrm{Tate}_{\mathfrak{b},\mathfrak{a}}}, \mathcal{O}_F \otimes \mathfrak{d}^{-1})$. Therefore by the first trivialisation (2.10),

$$\omega^\vee_{\mathrm{Tate}_{\mathfrak{b},\mathfrak{a}}/S_X} \simeq (\mathfrak{b} \otimes_{\mathbb{Z}} \mathcal{O}_{S_X})^\vee \simeq \mathfrak{b}^* \otimes_{\mathbb{Z}} \mathcal{O}_{S_X} \ .$$

Using (2.10) and the short exact sequence (2.12), one gets that

$$\wedge^2_{\mathcal{O}_F\otimes\mathcal{O}_{Y_\mathfrak{c}}} \mathcal{H}^1_{\mathrm{dR}}(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}/S_X) \simeq (\mathfrak{a}\mathfrak{b}^* \otimes_{\mathbb{Z}} \mathcal{O}_{S_X}) = \mathfrak{c}\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathcal{O}_{S_X} \ .$$

This finishes the proof. □

From now on, if not specified, tensor product will be taken over $\mathbb{Z}$.

Remark 2.3.4. Recall that at the beginning of Section 2.1.1, wa have decomposed the sheaf

$$\pi_*\Omega^1_{A/S} = \bigoplus_{\mathfrak{p}|p} \omega_{A/S,\mathfrak{p}} = \bigoplus_{\mathfrak{p}|p} \bigoplus_{j=1}^{f_\mathfrak{p}} \omega_{A/S,\mathfrak{p},j} \ ,$$

for any HBAS A over a scheme S. Applying this decomposition to the Tate object $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}$ over S_X, using the definition of the $\dot{\omega}_\tau$ for any $\tau = \tau_{\mathfrak{p},j}^{(i)}$, and by the trivializations in Equations (2.10) and (2.11), one has the following canonical identification

$$\dot{\omega}^{k,\ell}_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}/S_X} \overset{\mathrm{can}(\mathfrak{a},\mathfrak{b})}{=} (\mathfrak{a}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_X} , \tag{2.13}$$

where by $(\mathfrak{a}\otimes\mathcal{O})^k$ and $(\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell$ we mean the free $\mathcal{O}$-module of rank 1 defined as

$$(\mathfrak{a}\otimes\mathcal{O})^k := \bigotimes_{\tau\in\Sigma}(\mathfrak{a}\otimes\mathcal{O})_\tau^{\otimes k_\tau} \ , \text{ and } \ (\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell := \bigotimes_{\tau\in\Sigma}(\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O})_\tau^{\otimes \ell_\tau},$$

where $(\mathfrak{a}\otimes\mathcal{O})_\tau$ denotes the copy of $\mathcal{O}$ identified via the embedding τ. In particular, the coefficients of q-expansions will live in the rank one $\mathcal{O}$-module $(\mathfrak{a}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{a}\mathfrak{b}^*\otimes\mathcal{O})^\ell$, while the powers of q will be elements of the power series ring $\mathcal{O}_{S_X}$ (see Proposition 2.4.1).

We take now the time to describe the effects on the line bundle $\dot{\omega}^{k,\ell}$ of multiplying either $\mathfrak{a}$ or $\mathfrak{b}$ by $\mathfrak{p}$, i.e. the effects of isogenies between the corresponding Tate varieties. We will use these results later when computing the effect of the Hecke operator at $\mathfrak{p}$ on $\mathfrak{q}$-expansions.

Proposition 2.3.5. *Let $\mathfrak{p}$ be a prime in $\mathcal{O}_F$ above p. Let $\mathfrak{a}$ be a fractional ideal of F coprime with p. Then the natural inclusion $\mathfrak{a}\mathfrak{p}\hookrightarrow\mathfrak{a}$ induces a commutative diagram of $\mathcal{O}_{S_X}$-modules*

$$\begin{tikzcd}
\dot{\omega}^{k,\ell}_{\mathrm{Tate}_{\mathfrak{a}\mathfrak{p},\mathfrak{b}}/S_X} \arrow[r, hook] \arrow[d, equal, "\mathrm{can}(\mathfrak{a}\mathfrak{p},\mathfrak{b})"'] & \dot{\omega}^{k,\ell}_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}/S_X} \arrow[d, equal, "\mathrm{can}(\mathfrak{a},\mathfrak{b})"] \\
(\mathfrak{a}\mathfrak{p}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{a}\mathfrak{b}^*\mathfrak{p}\otimes\mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_X} \arrow[r, hook] & (\mathfrak{a}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{a}\mathfrak{b}^*\otimes\mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_X}
\end{tikzcd}$$

where $X\supseteq\mathfrak{a}\mathfrak{b}$.

Proof. Recall that the Tate object $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}$ is defined over any S_X, where $X\supseteq\mathfrak{a}\mathfrak{b}$ (see Definition 2.3.1). Now, since $\mathfrak{a}\mathfrak{p}\mathfrak{b}\subseteq\mathfrak{a}\mathfrak{b}$, the Tate varieties $\mathrm{Tate}_{\mathfrak{a}\mathfrak{p},\mathfrak{b}}$ and $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}$ can both be considered as S_X-schemes, for $X\supseteq\mathfrak{a}\mathfrak{b}$. The natural inclusion $\mathfrak{a}\mathfrak{p}\hookrightarrow\mathfrak{a}$ induces an S_X-isogeny on the associated tori

$$\mathbb{G}_m\otimes\mathfrak{a}^* \twoheadrightarrow \mathbb{G}_m\otimes(\mathfrak{a}\mathfrak{p})^* ,$$

which translates to an isogeny on the Tate varieties as S_X-schemes:

$$\begin{tikzcd}
\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}} \arrow[rr] \arrow[dr] & & \mathrm{Tate}_{\mathfrak{a}\mathfrak{p},\mathfrak{b}} \arrow[dl] \\
& S_X &
\end{tikzcd}$$

Since differential forms and the sheaf $\wedge^2_{\mathcal{O}_F\otimes\mathcal{O}_{Y_\mathfrak{c}}}\mathcal{H}^1_{\mathrm{dR}}$ are contravariant, and using the identification of Equation 2.13, the above isogeny induces an injective morphism of $\mathcal{O}_{S_X}$-modules

$$(\mathfrak{a}\mathfrak{p}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{a}\mathfrak{b}^*\mathfrak{p}\otimes\mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_X} \hookrightarrow (\mathfrak{a}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{a}\mathfrak{b}^*\otimes\mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_X} ,$$

which gives the desired result. $\square$

Proposition 2.3.6. *Let $\mathfrak{p}$ be a prime in $\mathcal{O}_F$ above p. Let $\mathfrak{b}$ be a fractional ideal of F coprime with p. Then the natural inclusion $\mathfrak{a}\mathfrak{b}^*\mathfrak{p} \hookrightarrow \mathfrak{a}\mathfrak{b}^*$ induces a commutative diagram of $\mathcal{O}_{S_X}$-modules*

$$\begin{tikzcd}
\dot{\omega}^{k,\ell}_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}\mathfrak{p}^{-1}}/S_X} \arrow[r, hook] \arrow[d, equal, "\mathrm{can}(\mathfrak{a},\mathfrak{b}\mathfrak{p}^{-1})"'] & \dot{\omega}^{k,\ell}_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}/S_X} \arrow[d, equal, "\mathrm{can}(\mathfrak{a},\mathfrak{b})"] \\
(\mathfrak{a}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{a}\mathfrak{b}^*\mathfrak{p}\otimes\mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_X} \arrow[r, hook] & (\mathfrak{a}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{a}\mathfrak{b}^*\otimes\mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_X}
\end{tikzcd}$$

where $X \supseteq \mathfrak{a}\mathfrak{b}\mathfrak{p}^{-1}$.

Proof. Again as in the previous proposition, since X contains $\mathfrak{a}\mathfrak{b}\mathfrak{p}^{-1}$ and $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a}\mathfrak{b}\mathfrak{p}^{-1}$, both Tate varieties are defined over S_X. Now, since $\mathfrak{b} \subset \mathfrak{b}\mathfrak{p}^{-1}$, one has an isogeny on the Tate varieties as S_X-schemes:

$$\begin{tikzcd}
\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}} = (\mathbb{G}_m\otimes\mathfrak{a}^*)/q(\mathfrak{b}) \arrow[rr] \arrow[dr] & & (\mathbb{G}_m\otimes\mathfrak{a}^*)/q(\mathfrak{b}\mathfrak{p}^{-1}) = \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}\mathfrak{p}^{-1}} \arrow[dl] \\
& S_X &
\end{tikzcd}$$

Note that by the canonical identification of Equation (2.13), on the $\dot{\omega}$ part of the sheaf we will have an isomorphism, since the above isogeny does not have an effect on the group of characters of $\mathfrak{a}$. However, it does have an effect on the periods, and using the equation (2.11), one gets the desired map via the natural inclusion $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{p} \hookrightarrow \mathfrak{a}\mathfrak{b}^{-1}$. □

2.4 q-Expansions Rings

We recall that $\mathfrak{C}$ is a fixed set of representatives of Cl_F^+. For every $\mathfrak{c} \in \mathfrak{C}$ we have a collection of $\mathfrak{c}$-cusps, $\mathcal{C} = (\mathfrak{a}, \mathfrak{b}, H, i, j, \lambda, \alpha)$, obtained by varying $\mathfrak{a}$ et $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$. We set $\infty(\mathfrak{c})$ to denote the standard $\mathfrak{c}$-cusp at infinity, i.e. the $\mathfrak{c}$-cusp where $\mathfrak{a} = \mathfrak{c}$ and $\mathfrak{b} = \mathcal{O}_F$. Moreover, for every cusp $\mathcal{C}$, we have a Tate object $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}$ over a scheme S_X, which depends on the smooth rational polyhedral admissible cone decomposition of the fractional ideal X, containing $\mathfrak{a}\mathfrak{b}$. We will use these ingredients to make explicit the local completed module of the sheaf $\omega^{k,\ell}$ over $\mathrm{Sh}^{\mathrm{tor}}$ along the cusp $\mathcal{C}$.

Proposition 2.4.1. *For a $\mathfrak{c}$-cusp, $\mathcal{C} = (\mathfrak{a}, \mathfrak{b}, H, i, j, \lambda, \alpha)$ with associated fractional ideal $X \supset \mathfrak{a}\mathfrak{b}$, the completion of $\omega^{k,\ell}$ over $\mathrm{Sh}^{\mathrm{tor}}$ at $\mathcal{C}$ is given by*

$$\mathcal{M}^{k,\ell}_{\mathfrak{a},\mathfrak{b}}(X) := \left\{ \sum_{\xi\in X_+\cup\{0\}} a_\xi q^\xi \;\middle|\; a_\xi \in \left(\mathfrak{a}\otimes\mathcal{O}\right)^k \otimes_{\mathcal{O}} \left(\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O}\right)^\ell;\ a_{\varepsilon\xi} = \varepsilon^{-\ell} a_\xi \textit{ for all } \varepsilon \in \mathcal{O}_{F,+}^\times \right\}. \tag{2.14}$$

For the infinity cusp $\infty(\mathfrak{c})$, we will denote by $\mathcal{M}^{k,\ell}_\infty(\mathfrak{c}) := \mathcal{M}^{k,\ell}_{\mathfrak{c},\mathcal{O}_F}(\mathfrak{c})$ and we will call it the module of q-expansions at the cusp $\infty(\mathfrak{c})$.

We remark that the above description of the module of q-expansion agrees with the one given by Diamond and Sasaki ([DS17, Proposition 9.1.2]), where they work with the Deligne-Pappas model since they are assuming that p is unramified in F.

Proof. In this proof we will use the notation and results of Dimitrov, in [Dim04]. Let $\mathcal{C} = (\mathfrak{a}, \mathfrak{b}, H, i, j, \lambda, \alpha)$ be a $\mathfrak{c}$-cusp and $\Sigma^{\mathcal{C}}$ be a smooth rational polyhedral admissible cone decomposition of X_+^*. We will be working in a formal neighborhood of the cusp $\mathcal{C}$, given by the formal completed scheme $S^{\wedge}_{\Sigma^{\mathcal{C}}}$, which is the completion of the variety obtained by gluing all the toric immersions at infinity. By [Dim04, Théorème 8.6 (v)], we know that the formal completion of $\mathrm{Sh}^{\mathrm{tor}}$ along the cusp $\mathcal{C}$ is canonically isomorphic to $S^{\wedge}_{\Sigma^{\mathcal{C}}}/\mathcal{O}^{\times}_{F,\mathfrak{n}} \times \mathcal{O}^{\times}_{F,+}$, and in particular the completion of $\omega^{k,\ell}$ over $\mathrm{Sh}^{\mathrm{tor}}$ at $\mathcal{C}$ can be identified with the global sections $\mathrm{H}^0(S^{\wedge}_{\Sigma^{\mathcal{C}}}/\mathcal{O}^{\times}_{F,\mathfrak{n}} \times \mathcal{O}^{\times}_{F,+}, \omega^{k,\ell})$. Moreover, one can determine the set of global sections $\mathrm{H}^0(S^{\wedge}_{\Sigma^{\mathcal{C}}}/\mathcal{O}^{\times}_{F,\mathfrak{n}} \times \mathcal{O}^{\times}_{F,+}, \omega^{k,\ell})$ by taking invariants under the action of $\mathcal{O}^{\times}_{F,\mathfrak{n}} \times \mathcal{O}^{\times}_{F,+}$ of the set of global sections $\mathrm{H}^0(S^{\wedge}_{\Sigma^{\mathcal{C}}}, \dot{\omega}^{k,\ell})$. As explained in Remark 2.3.4, the module $\mathrm{H}^0(S^{\wedge}_{\Sigma^{\mathcal{C}}}, \dot{\omega}^{k,\ell})$ can be described using the trivialisations (2.10) and (2.11), as

$$\dot{\omega}^{k,\ell}_{|S^{\wedge}_{\Sigma^{\mathcal{C}}}} \overset{\mathrm{can}(\mathfrak{a},\mathfrak{b})}{=} (\mathfrak{a} \otimes_{\mathbb{Z}} \mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathcal{O})^{\ell} \otimes_{\mathcal{O}} \mathcal{O}_{S^{\wedge}_{\Sigma^{\mathcal{C}}}} .$$

An element in $\mathrm{H}^0(S^{\wedge}_{\Sigma^{\mathcal{C}}}, \dot{\omega}^{k,\ell})$ is then a power series $\sum_{\xi \in X_+ \cup \{0\}} a_\xi q^\xi$, with $a_\xi \in (\mathfrak{a} \otimes_{\mathbb{Z}} \mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mathcal{O})^{\ell}$, which we recall is a free of rank one $\mathcal{O}$-module. Now, let us recall that the group $\mathcal{O}^{\times}_{F,\mathfrak{n}} \times \mathcal{O}^{\times}_{F,+}$ acts on the cusp $\mathcal{C}$ via a matrix $\begin{pmatrix} \varepsilon u & 0 \\ 0 & u^{-1} \end{pmatrix}$ on the $\mathcal{O}_F$-lattice $H \simeq \mathfrak{b} \oplus \mathfrak{a}^*$, as in [Dim04, Proposition 3.3]. In particular, under this action, one has that $\mathfrak{a} \mapsto u\mathfrak{a}$ and $\mathfrak{b} \mapsto \varepsilon u \mathfrak{b}$; therefore we also have that $\mathfrak{c} \mapsto \varepsilon^{-1}\mathfrak{c}$ and $X \mapsto (\varepsilon u^2)X$. Therefore the Fourier coefficients must satisfy the following

$$a_{(u^2\varepsilon)\xi} = u^k \varepsilon^{-\ell} a_\xi .$$

The q-expansion ring for $\mathcal{C}$ is identified with

$$\mathrm{H}^0(S^{\wedge}_{\Sigma^{\mathcal{C}}}/\mathcal{O}^{\times}_{F,\mathfrak{n}} \times \mathcal{O}^{\times}_{F,+},\ \omega^{k,\ell}) = \left\{ \sum_{\xi \in X_+ \cup \{0\}} a_\xi q^\xi \;\middle|\; \begin{array}{l} a_\xi \in (\mathfrak{a} \otimes \mathcal{O})^k \otimes (\mathfrak{c}\mathfrak{d}^{-1} \otimes \mathcal{O})^{\ell}; \\ a_{(u^2\varepsilon)\xi} = u^k \varepsilon^{-\ell} a_\xi \text{ for all } \varepsilon \in \mathcal{O}^{\times}_{F,+}, u \in \mathcal{O}^{\times}_{F,\mathfrak{n}} \end{array} \right\} .$$

For a scalar matrix, i.e. for $\varepsilon = u^{-2}$, the action on X is then trivial and in fact under the Hypothesis 5, one has

$$a_{(u^2\varepsilon)\xi} = u^k \varepsilon^{-\ell} a_\xi = u^{k+2\ell} a_\xi = a_\xi .$$

Since scalar matrices act trivially, one can decompose the matrix $\begin{pmatrix} \varepsilon u & 0 \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} \cdot \begin{pmatrix} \varepsilon u^2 & 0 \\ 0 & 1 \end{pmatrix}$, and just look at the action of matrices of the form $\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$ for $\varepsilon \in \mathcal{O}^{\times}_{F,+}$, which gives that $a_{\varepsilon\xi} = \varepsilon^{\ell} a_\xi$ and therefore the Proposition. □

Corollary 2.4.2. *Let R be a $\mathcal{O}$-algebra satisfying Hypothesis 5. Then for a $\mathfrak{c}$-cusp, $\mathcal{C} = (\mathfrak{a}, \mathfrak{b}, H, i, j, \lambda, \alpha)$ with associated fractional ideal $X \supset \mathfrak{a}\mathfrak{b}$, the completion of $\omega^{k,\ell}_R$ over $\mathrm{Sh}^{\mathrm{tor}}_R$ at $\mathcal{C}$ is given by*

$$\mathcal{M}^{k,\ell}_{\mathfrak{a},\mathfrak{b}}(X; R) := \left\{ \sum_{\xi \in X_+ \cup \{0\}} a_\xi q^\xi \;\middle|\; \begin{array}{l} a_\xi \in \left((\mathfrak{a} \otimes \mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}\mathfrak{d}^{-1} \otimes \mathcal{O})^{\ell} \right) \otimes_{\mathcal{O}} R; \\ a_{\varepsilon\xi} = \varepsilon^{-\ell} a_\xi \textit{ for all } \varepsilon \in \mathcal{O}^{\times}_{F,+} \end{array} \right\} .$$

Proof. This follows immediatly from Proposition 2.4.1, and by the canonical identification

$$\dot{\omega}^{k,\ell}_{|S^{\wedge}_{\Sigma^{\mathcal{C}}}\times \mathrm{Spec}(R)} \overset{\mathrm{can}(\mathfrak{a},\mathfrak{b})}{=} \left((\mathfrak{a}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O})^{\ell}\right) \otimes_{\mathcal{O}} R \otimes_{\mathcal{O}} \mathcal{O}_{S^{\wedge}_{\Sigma^{\mathcal{C}}}\times\mathrm{Spec}(R)}\ .$$

induced by Equation (2.13). □

We will mainly consider expansion rings for the standard cusps at infinity, $\infty(\mathfrak{c})$. In particular one has injective q-expansion maps

$$\mathrm{H}^0(\mathrm{Sh}^{\mathrm{tor}},\omega^{k,\ell}) \hookrightarrow \bigoplus_{\mathfrak{c}\in\mathfrak{C}} \mathcal{M}^{k,\ell}_{\infty}(\mathfrak{c}) \ \text{ and } \ \mathrm{H}^0(\mathrm{Sh}^{\mathrm{tor}}_R,\omega^{k,\ell}_R) \hookrightarrow \bigoplus_{\mathfrak{c}\in\mathfrak{C}} \mathcal{M}^{k,\ell}_{\infty}(\mathfrak{c};R).$$

Since we are working over $\mathcal{O}$, it will be very important for our computations on the q-expansion to keep working with the set $\mathfrak{C}$ of representatives coprime with p. However, we will have to manipulate as seen in the prevision section, the $\mathfrak{p}$-isogenies on the Tate objects (see Proposition 2.3.5). We then end this chapter by showing what happens on q-expansions when we bring a cusp to the fixed set of representatives $\mathfrak{C}$.

Lemma 2.4.3. *Let $\mathfrak{c}\in\mathfrak{C}$ and $\mathfrak{p}$ a prime above p. Let $\alpha\in F_+$ such that $\mathfrak{c}\mathfrak{p}=\alpha\mathfrak{c}'$, where $\mathfrak{c}'\in\mathfrak{C}$ is a representative in Cl^+_F. We have the following morphism of $\mathcal{O}$-modules:*

$$\begin{aligned} \mathcal{M}^{k,\ell}_{\infty}(\mathfrak{c}') &\xrightarrow{\cdot\alpha^{k+\ell}} \mathcal{M}^{k,\ell}_{\infty}(\mathfrak{c}\mathfrak{p}) \\ \sum_{\xi'\in\mathfrak{c}'_+\cup\{0\}} a_{\xi'}q^{\xi'} &\longmapsto \sum_{\xi'\in\mathfrak{c}'_+\cup\{0\}} \alpha^{k+\ell}a_{\xi'}q^{\alpha\xi'} = \sum_{\xi\in(\mathfrak{c}\mathfrak{p})_+\cup\{0\}} \alpha^{k+\ell}a_{\alpha^{-1}\xi}q^{\xi}. \end{aligned}$$

Proof. In order to establish the morphism above, one has to look at the cusps $\infty(\mathfrak{c}\mathfrak{p})$ and $\infty(\mathfrak{c}')$, and their associated Tate varieties. But first let us recall that the scheme S_X is constructed from the scheme $\mathrm{Spec}(\mathcal{O}[[q^{\xi}:\xi\in X_+]])$, and that in particular this construction based on the smooth rational polyhedral cone decomposition is functorial (see [Dim04, Section 2]). In particular, the base sheaves $S_{\mathfrak{c}\mathfrak{p}}$ and $S_{\mathfrak{c}'}$ are isomorphic, where the isomorphism is induced by the ring isomorphism

$$\begin{aligned} \mathcal{O}[q^{\xi'}:\xi'\in\mathfrak{c}'] &\xrightarrow{\sim} \mathcal{O}[q^{\xi}:\xi\in\mathfrak{c}\mathfrak{p}]\ , \\ q^{\xi'} &\longmapsto q^{\alpha\xi'} \\ q^{\alpha^{-1}\xi} &\longleftarrow q^{\xi}\ . \end{aligned}$$

We can then see the Tate variety $\mathrm{Tate}_{\mathfrak{c}\mathfrak{p},\mathcal{O}_F}$ as a variety over $S_{\mathfrak{c}'}$ via pullback in the following cartesian square

$$\begin{tikzcd} \mathrm{Tate}_{\mathfrak{c}\mathfrak{p},\mathcal{O}_F} \arrow[r] \arrow[d] & (\mathbb{G}_m\otimes(\mathfrak{c}\mathfrak{p})^*)/q(\mathcal{O}_F) \arrow[d] \\ S_{\mathfrak{c}'} \arrow[r, "\sim"] & S_{\mathfrak{c}\mathfrak{p}} \end{tikzcd}$$

i.e. we can see $\mathrm{Tate}_{\mathfrak{c}\mathfrak{p},\mathcal{O}_F}$ as the $S_{\mathfrak{c}'}$-scheme $\left((\mathbb{G}_m\otimes(\mathfrak{c}\mathfrak{p})^*)/q(\mathcal{O}_F)\right)\times_{S_{\mathfrak{c}\mathfrak{p}}} S_{\mathfrak{c}'}$.[6] We can then look

[6] Concretely, we are re-indexing the powers of q in the q-expansion.

at both Tate varieties as $S_{\mathfrak{c}'}$-schemes and consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_F \arrow[r] \arrow[d, equal] & \mathbb{G}_m \otimes (\mathfrak{c}\mathfrak{p})^* \arrow[r] \arrow[d, "1\otimes\alpha"] & \mathbb{G}_m \otimes (\mathfrak{c}\mathfrak{p})^*/q(\mathcal{O}_F) \arrow[rr] \arrow[d, "1\otimes\alpha"] \arrow[dr] & & 0 \\
& & & & S_{\mathfrak{c}'} & \\
0 \arrow[r] & \mathcal{O}_F \arrow[r] & \mathbb{G}_m \otimes (\mathfrak{c}')^* \arrow[r] & \mathbb{G}_m \otimes (\mathfrak{c}')^*/q(\mathcal{O}_F) \arrow[rr] \arrow[ur] & & 0
\end{tikzcd}$$

Recall that ω_τ and δ_τ are contravariant, and that by the canonical identification in (2.13), the above diagram induces a morphism:

$$(\mathfrak{c}' \otimes \mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}'\mathfrak{d}^{-1} \otimes \mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_{\mathfrak{c}'}} \to (\mathfrak{c}\mathfrak{p} \otimes \mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}\mathfrak{p}\mathfrak{d}^{-1} \otimes \mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_{\mathfrak{c}'}} \tag{2.15}$$

For every embedding τ, one has the following commutative diagram of morphisms of $\mathcal{O}$-modules:

$$\begin{tikzcd}
(\mathfrak{c}' \otimes \mathcal{O})_\tau \arrow[r] \arrow[d, "\cdot\alpha"] & \mathrm{Frac}(\mathcal{O}) = K \arrow[d, "\tau(\alpha)"] \\
(\mathfrak{c}\mathfrak{p} \otimes \mathcal{O})_\tau \arrow[r] & \mathrm{Frac}(\mathcal{O}) = K
\end{tikzcd}
\qquad
\begin{tikzcd}
(\xi' \otimes 1) \arrow[r, mapsto] \arrow[d, mapsto, "\cdot\alpha"] & \tau(\xi') \arrow[d, mapsto, "\cdot\tau(\alpha)"] \\
(\alpha\xi' \otimes 1) \arrow[r, mapsto] & \tau(\alpha\xi')
\end{tikzcd}$$

which give morphisms on the $\mathcal{O}$-modules:

$$(\mathfrak{c}' \otimes \mathcal{O})^k \xrightarrow{\cdot\alpha^k} (\mathfrak{c}\mathfrak{p} \otimes \mathcal{O})^k \qquad\qquad (\mathfrak{c}'\mathfrak{d}^{-1} \otimes \mathcal{O})^\ell \xrightarrow{\cdot\alpha^\ell} (\mathfrak{c}\mathfrak{p}\mathfrak{d}^{-1} \otimes \mathcal{O})^\ell$$

So putting the two together, for $\xi' \in \mathfrak{c}'$ one gets a morphism of the modules of coefficients,

$$\begin{gathered}
(\mathfrak{c}' \otimes \mathcal{O})^k \otimes (\mathfrak{c}'\mathfrak{d}^{-1} \otimes \mathcal{O})^\ell \to (\mathfrak{c}\mathfrak{p} \otimes \mathcal{O})^k \otimes (\mathfrak{c}\mathfrak{p}\mathfrak{d}^{-1} \otimes \mathcal{O})^\ell \\
a_{\xi'} \mapsto \alpha^{k+\ell} a_{\xi'} \ .
\end{gathered}$$

Now using the morphism on the coefficients and Equation (2.15), one obtains the following morphism of $\mathcal{O}$-modules

$$\begin{aligned}
\mathcal{M}_\infty^{k,\ell}(\mathfrak{c}') &\xrightarrow{\cdot\alpha^{k+\ell}} \mathcal{M}_\infty^{k,\ell}(\mathfrak{c}\mathfrak{p}) \\
\sum_{\xi' \in \mathfrak{c}'_+ \cup \{0\}} a_{\xi'} q^{\xi'} &\longmapsto \sum_{\xi' \in \mathfrak{c}'_+ \cup \{0\}} \alpha^{k+\ell} a_{\xi'} q^{\alpha\xi'} = \sum_{\xi \in (\mathfrak{c}\mathfrak{p})_+ \cup \{0\}} b_\xi q^\xi
\end{aligned}$$

where $b_\xi = \alpha^{k+\ell} a_{\alpha^{-1}\xi} \in (\mathfrak{c}\mathfrak{p} \otimes \mathcal{O})^k \otimes (\mathfrak{c}\mathfrak{p}\mathfrak{d}^{-1} \otimes \mathcal{O})^\ell$. In fact, this morphism respects the conditions given by the $\mathcal{O}_{F,+}^\times$-action, as in Proposition 2.4.1: for $\varepsilon \in \mathcal{O}_{F,+}^\times$

$$b_{\varepsilon\xi} = \alpha^{k+\ell} a_{\varepsilon\alpha^{-1}\xi} = \alpha^{k+\ell} a_{\varepsilon\xi'} = \varepsilon^{-\ell} \alpha^{k+\ell} a_{\xi'} = \varepsilon^{-\ell} b_\xi \ ,$$

where the one before last equality is given by the fact that $a_{\xi'}$ satisfies $a_{\varepsilon\xi'} = \varepsilon^{-\ell} a_{\xi'}$ for all $\varepsilon \in \mathcal{O}_{F,+}^\times$ and $\xi' \in \mathfrak{c}'$. $\square$

Remark 2.4.4. We remark that this is not an isomorphism because α has strictly positive $\mathfrak{p}$-adic valuation. In fact $v_{\mathfrak{p}}(\alpha) = 1$, since $\alpha\mathfrak{c}' = \mathfrak{c}\mathfrak{p}$, and $\mathfrak{c}, \mathfrak{c}' \in \mathfrak{C}$ are coprime with p.

The strategy of proof of Lemma 2.4.3 can be applied to see what happens on the q-expansion when changing the representative of the class $\mathfrak{c} \in \mathfrak{C}$.

Lemma 2.4.5. *Let $\mathfrak{c}$ and $\nu\mathfrak{c}$, with $\nu \in F_+^\times$, be two representatives of the same ideal class group element, both coprime with p. Then we have an isomorphism*

$$\begin{aligned} \mathcal{M}_\infty^{k,\ell}(\mathfrak{c}) &\xrightarrow[\sim]{\cdot \nu^{k+\ell}} \mathcal{M}_\infty^{k,\ell}(\nu\mathfrak{c}) \\ \sum_{\xi \in \mathfrak{c}_+ \cup \{0\}} a_\xi q^\xi &\longmapsto \sum_{\xi \in \mathfrak{c}_+ \cup \{0\}} \nu^{k+\ell} a_\xi q^{\nu\xi} \end{aligned}$$

Proof. The proof goes exactly like in Lemma 2.4.3. However, this time one has an isomorphism since ν is invertible in $\mathcal{O}$, because $\nu\mathfrak{c}$ is assumed to be coprime with p. □

Chapter 3

Action of the $T_{\mathfrak{p}}$ operator on q-expansions

In this chapter, we construct step by step the Hecke operator at $\mathfrak{p}$, for a prime $\mathfrak{p} \subset \mathcal{O}_F$ dividing p, acting on Hilbert modular forms and compute its action on geometric q-expansions. The Hecke operator at $\mathfrak{p}$ in characteristic p was first constructed for paritious weights by Emerton, Reduzzi and Xiao (see [ERX17a, Section 3]). In particular, they construct a normalized Hecke operator at $\mathfrak{p}$ (see [ERX17a, Definition 3.12]) that acts on the whole cohomology $\mathrm{H}^\bullet(\mathrm{Sh}^{\mathrm{PR,tor}}_{\mathcal{O}/\varpi^m\mathcal{O}}, \omega^{k,\ell}_{\mathcal{O}/\varpi^m\mathcal{O}})$ in positive characteristic. We will only be interested in the degree 0 cohomology, and we will use their construction alongside some techniques of Dimitrov-Wiese (see [DW18, Section 3.3]) to calculate the action of the Hecke operator at $\mathfrak{p}$ on q-expansions for a generic partial weight (k, ℓ) as in Chapter 2.

The geometric construction of Emerton, Reduzzi and Xiao ([ERX17a]) gives rise to a Hecke operator at $\mathfrak{p}$ that we will denote $T_{\mathfrak{p}}^{\vee,\circ}$. The $\circ$ in this notation is to recall that this Hecke operator is normalized in order for it to be optimally integral on $\mathcal{O}$, and therefore to give rise to a non-trivial operator modulo ϖ. The dual is due to the fact that we are later interested to work with Galois representations attached to Hilbert modular forms, where the dual operator of the one constructed in [ERX17a] intervenes. The action of $T_{\mathfrak{p}}^{\vee,\circ}$ on q-expansions is given in Theorem 3.3.4.

In general, $T_{\mathfrak{p}} = T_{\mathfrak{p}}^{\vee} \circ \langle \mathfrak{p} \rangle$, where $\langle \mathfrak{p} \rangle$ denotes the diamond operator at $\mathfrak{p}$. However, the classical diamond operators, even if they come from a natural construction, do not give rise to an automorphism over $\mathcal{O}$ for primes dividing p. One can overcome this issue by working with paritious weights, i.e. weights $(k, \ell) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$ such that $k_\tau + 2\ell_\tau = \mathsf{w} \in \mathbb{Z}$ for all $\tau \in \Sigma$, and by normalizing by $\mathrm{Nm}(\mathfrak{p})^{\mathsf{w}}$ (see Definition 3.1.1). We therefore set $T_{\mathfrak{p}}^{\circ} := T_{\mathfrak{p}}^{\vee,\circ} \circ \langle \mathfrak{p} \rangle_{\mathsf{w}}$, where $\langle \mathfrak{p} \rangle_{\mathsf{w}}$ denotes the normalized diamond operator at $\mathfrak{p}$, and we compute its action on q-expansion in Corollary 3.3.6.

Finally, we recall to the reader that we are not imposing p to be unramified in $\mathcal{O}_F$, and therefore we are working with the Pappas-Rapoport model. In particular, from now on, we will drop the PR from all notations.

3.1 Normalized diamond operators

As explained in the introduction, we want to work with normalized diamond operators. We will recall here how the general diamond operator for a prime ideal $\mathfrak{q} \subset \mathcal{O}_F$ not dividing $\mathfrak{n}$ is constructed and we will explain why the normalization is essential to have an automophism on $\mathrm{H}^0(\mathrm{Sh}, \omega^{k,\ell})$, for paritious weights $(k,\ell) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$, i.e. such that $k_\tau + 2\ell_\tau = \mathsf{w} \in \mathbb{Z}$ for all $\tau \in \Sigma$. Let A be a HBAS over S, and let $\mathfrak{q}$ be a prime ideal of $\mathcal{O}_F$. Let us consider the following exact sequence of HBAS over S:

$$0 \to A[\mathfrak{q}] \to A \to A \otimes_{\mathcal{O}_F} \mathfrak{q}^{-1} \to 0, \tag{3.1}$$

where $A \otimes_{\mathcal{O}_F} \mathfrak{q}^{-1}$ denotes the S-scheme representing the functor of points as in Section 2.1.1. Since the Cartier dual of $A[\mathfrak{q}]$ is $(A \otimes_{\mathcal{O}_F} \mathfrak{q}^{-1})^\vee[\mathfrak{q}]$ (see [Hid04, Section 4.1.9]), dualizing the short exact sequence (3.1) gives the following short exact sequence of HBAS over S:

$$0 \to (A \otimes_{\mathcal{O}_F} \mathfrak{q}^{-1})^\vee[\mathfrak{q}] \to (A \otimes_{\mathcal{O}_F} \mathfrak{q}^{-1})^\vee \to A^\vee \to 0. \tag{3.2}$$

Let us now suppose that A is $\mathfrak{c}$-polarized, with polarization λ. It then results from the natural S-isogeny $A \to A \otimes_{\mathcal{O}_F} \mathfrak{q}^{-1}$ and the short exact sequence (3.2) that there is a canonical isomorphism (see [DW18, Equation 10])

$$(A \otimes_{\mathcal{O}_F} \mathfrak{q}^{-1})^\vee \xrightarrow{\sim} A^\vee \otimes_{\mathcal{O}_F} \mathfrak{q}. \tag{3.3}$$

Let $\mathfrak{c}' \in \mathfrak{C}$ and $\theta \in F_+$ such that $\theta\mathfrak{c}' = \mathfrak{c}\mathfrak{q}^2$. Using the above equation, one sees that the HBAS $A \otimes \mathfrak{q}^{-1}$ admits a $\mathfrak{c}'$-polarization:

$$\lambda' : (A \otimes_{\mathcal{O}_F} \mathfrak{q}^{-1}) \otimes_{\mathcal{O}_F} \mathfrak{c}' \xrightarrow[\sim]{1\otimes\theta} A \otimes_{\mathcal{O}_F} \mathfrak{c}\mathfrak{q} \xrightarrow[\sim]{\lambda} A^\vee \otimes \mathfrak{q} \xrightarrow[3.3]{\sim} (A \otimes_{\mathcal{O}_F} \mathfrak{q}^{-1})^\vee \tag{3.4}$$

(This can also be seen in [ERX17a, Section 2.9]). We then consider the isomorphism

$$\begin{aligned} \varphi_{\mathfrak{q}} : \mathrm{Y}_{\mathfrak{c}} &\longrightarrow \mathrm{Y}_{\mathfrak{c}'} \\ (A, \lambda, \mu, \mathcal{F}) &\longmapsto (A' := A \otimes_{\mathcal{O}_F} \mathfrak{q}^{-1}, \lambda', \mu', \mathcal{F}'), \end{aligned}$$

where λ' is the $\mathfrak{c}'$-polarization given in Equation (3.4) , μ' is the $\mu_{\mathfrak{n}}$-structure induced by μ and $(A \otimes_{\mathcal{O}_F} \mathfrak{q}^{-1})[\mathfrak{n}] \simeq A[\mathfrak{n}]$ (for primes $\mathfrak{q}$ coprime with $\mathfrak{n}$), and $\mathcal{F}'$ is induced filtration. This isomorphism extends to an isomorphism on the toroidal compactifications $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{tor}} \xrightarrow{\sim} \mathrm{Y}_{\mathfrak{c}'}^{\mathrm{tor}}$, by sending a $\mathfrak{c}$-cusp $\mathcal{C} = (\mathfrak{a}, \mathfrak{b}, H, i, j, \Lambda, \gamma)$ to the $\mathfrak{c}'$-cusp $\mathcal{C}' = (\mathfrak{a}\mathfrak{q}, \theta\mathfrak{b}\mathfrak{q}^{-1}, H \otimes \mathfrak{q}^{-1}, i \otimes \mathfrak{q}^{-1}, j \otimes \theta\mathfrak{q}^{-1}, \theta\Lambda, \gamma')$, where γ' is the obvious induced level structure (see discussion in [ERX17a, Section 2.9]).

Let us now see what happens on the sheaves. Let $\mathcal{A}_{\mathfrak{c}}$ and $\mathcal{A}_{\mathfrak{c}'}$ denote the universal semi-abelian varieties respectively over $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{tor}}$ and $\mathrm{Y}_{\mathfrak{c}'}^{\mathrm{tor}}$. Then one has the following commutative diagram of $\mathcal{O}$-schemes:

$$\begin{tikzcd} \mathcal{A}_{\mathfrak{c}} \arrow[r] \arrow[d] & \mathcal{A}_{\mathfrak{c}} \otimes_{\mathcal{O}_F} \mathfrak{q}^{-1} \arrow[r] \arrow[d] & \mathcal{A}_{\mathfrak{c}'} \arrow[dl] \\ \mathrm{Y}_{\mathfrak{c}}^{\mathrm{tor}} \arrow[r, "\varphi_{\mathfrak{q}}", "\sim"'] & \mathrm{Y}_{\mathfrak{c}'}^{\mathrm{tor}} & \end{tikzcd}$$

which induces a natural pullback morphism

$$\dot{S}_{\mathfrak{q}}^\vee : \varphi_{\mathfrak{q}}^* \dot{\omega}^{\mathrm{tor}}_{\mathcal{A}_{\mathfrak{c}'}/\mathrm{Y}_{\mathfrak{c}'}^{\mathrm{tor}}} \to \dot{\omega}^{\mathrm{tor}}_{\mathcal{A}_{\mathfrak{c}}/\mathrm{Y}_{\mathfrak{c}}^{\mathrm{tor}}},$$

where we recall that $\dot{\omega}^{\mathrm{tor}}_{\mathcal{A}_{\mathfrak{c}}/\mathrm{Y}_{\mathfrak{c}}^{\mathrm{tor}}}$ denotes the sheaf of relative differentials over $\mathrm{Y}_{\mathfrak{c}}^{\mathrm{tor}}$, constructed in Section 2.2.1 from $e^*\Omega^1_{\mathcal{A}_{\mathfrak{c}}^{\mathrm{R}}/\mathrm{Y}_{\mathfrak{c}}^{\mathrm{R,tor}}}$. We recall that the dot in the notation is used to recall that we are working over the moduli space Y, and not over the corresponding Shimura variety Sh. The dual in the notation is again due to the fact that this operator turns out to be the inverse of the

classical diamond operator.
Similarly, one has a natural morphism $\dot{S}_{\mathfrak{q}}^{\vee} : \varphi_{\mathfrak{q}}^{*}\dot{\delta}_{\mathfrak{c}',\tau} \to \dot{\delta}_{\mathfrak{c},\tau}$ for any $\tau \in \Sigma$. Let $(k,\ell) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}^{\Sigma}$ be a paritious weight, with $k_\tau + 2\ell_\tau = \mathsf{w}$ for all $\tau \in \Sigma$. One then has a natural isomorphism

$$\dot{S}_{\mathfrak{q}}^{\vee,\circ} : \varphi_{\mathfrak{q}}^{*}\left(\bigotimes_{\tau\in\Sigma} \dot{\omega}_{\mathfrak{c}',\tau}^{k_\tau} \otimes_{\mathcal{O}_{\mathrm{Y}_{\mathfrak{c}'}^{\mathrm{tor}}}} \dot{\delta}_{\mathfrak{c}',\tau}^{\ell_\tau}\right) \xrightarrow[\sim]{(\mathrm{Nm}\,\mathfrak{q})^{-\mathsf{w}}\dot{S}_{\mathfrak{q}}^{\vee}} \bigotimes_{\tau\in\Sigma} \dot{\omega}_{\mathfrak{c},\tau}^{k_\tau} \otimes_{\mathcal{O}_{\mathrm{Y}_{\mathfrak{c}}^{\mathrm{tor}}}} \dot{\delta}_{\mathfrak{c},\tau}^{\ell_\tau}.$$

We will explain in more details below why this is isomorphism. Taking the union over all $\mathfrak{c} \in \mathfrak{C}$, gives an isomorphism

$$\dot{S}_{\mathfrak{q}}^{\vee,\circ} : \mathrm{H}^0(\mathrm{Y}_{\mathfrak{c}'}^{\mathrm{tor}}, \dot{\omega}^{k,\ell}) \xrightarrow[\sim]{(\mathrm{Nm}\,\mathfrak{q})^{-\mathsf{w}}\dot{S}_{\mathfrak{q}}^{\vee}} \mathrm{H}^0(\mathrm{Y}_{\mathfrak{c}}^{\mathrm{tor}}, \dot{\omega}^{k,\ell}).$$

Moreover, this isomorphism passes to the quotient by the action of the group E (see [ERX17a, Section 2.9]), giving rise to

$$S_{\mathfrak{q}}^{\vee,\circ} : \mathrm{H}^0(\mathrm{Sh}^{\mathrm{tor}}, \omega^{k,\ell}) \xrightarrow[\sim]{(\mathrm{Nm}\,\mathfrak{q})^{-\mathsf{w}}S_{\mathfrak{q}}^{\vee}} \mathrm{H}^0(\mathrm{Sh}^{\mathrm{tor}}, \omega^{k,\ell}), \tag{3.5}$$

by taking the disjoint union over the fixed set of representatives $\mathfrak{C}$. Moreover, the action of $S_{\mathfrak{q}}^{\vee,\circ}$ is indipendent of the choice of the element θ (see [ERX17a, Section 2.9]).

Definition 3.1.1. Let $(k,\ell) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}^{\Sigma}$ be a paritious weight, with $k_\tau + 2\ell_\tau = \mathsf{w}$ for all $\tau \in \Sigma$. Let R be any $\mathcal{O}$-algebra. We then define the *diamond operator*

$$\langle \mathfrak{q} \rangle_{\mathsf{w}} : \mathrm{H}^0(\mathrm{Sh}_R^{\mathrm{tor}}, \dot{\omega}^{k,\ell}) \to \mathrm{H}^0(\mathrm{Sh}_R^{\mathrm{tor}}, \dot{\omega}_R^{k,\ell})$$

to be $\langle \mathfrak{q} \rangle_{\mathsf{w}} := \left(\mathrm{Nm}(\mathfrak{q})^{-\mathsf{w}} S_{\mathfrak{q}}^{\vee}\right)^{-1}$, the inverse of the induced properly normalized isomorphism of Equation 3.5.

We now explain why the normalization is essential by looking at what happens at the cusps. Let us recall that the Tate object at the $\mathfrak{c}$-cusp $\mathcal{C} = (\mathfrak{a}, \mathfrak{b}, H, i, j, \Lambda, \gamma)$ is $\mathbb{G}_m \otimes \mathfrak{a}^*/q(\mathfrak{b})$ over the base S_X (see Section 2.3). The isomorphism $\varphi_{\mathfrak{q}}$ send the $\mathfrak{c}$-cusp $\mathcal{C}$ to a $\mathfrak{c}\mathfrak{q}^2$-cusp $\mathcal{C}' = (\mathfrak{a}\mathfrak{q}, \mathfrak{b}\mathfrak{q}^{-1}, H \otimes \mathfrak{q}^{-1}, i \otimes \mathfrak{q}^{-1}, j \otimes \mathfrak{q}^{-1}, \Lambda, \gamma')$, inducing the following commutative diagram of S_X schemes[1]:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{b} \arrow[r, "q"] \arrow[d, hook] & \mathbb{G}_m \otimes \mathfrak{a}^* \arrow[r] \arrow[d, "1\otimes\mathfrak{q}^{-1}"] & \mathbb{G}_m \otimes \mathfrak{a}^*/q(\mathfrak{b}) \arrow[r] \arrow[d, "1\otimes\mathfrak{q}^{-1}"] \arrow[dr] & 0 \\
& & & & S_X \\
0 \arrow[r] & \mathfrak{b}\mathfrak{q}^{-1} \arrow[r, "q"] & \mathbb{G}_m \otimes (\mathfrak{a}\mathfrak{q})^* \arrow[r] & \mathbb{G}_m \otimes (\mathfrak{a}\mathfrak{q})^*/q(\mathfrak{b}\mathfrak{q}^{-1}) \arrow[r] \arrow[ur] & 0
\end{tikzcd}$$

where the second and third vertical maps are induced by the natural S_X-isogeny from Equation (3.1). Taking the $\mathfrak{c}$-cusp $\mathcal{C}$ to be the standard cusp at infinity $\infty(\mathfrak{c})$, this implies that there is a morphism of $S_{\mathfrak{c}}$-schemes:

$$\mathrm{Tate}_{\mathfrak{c},\mathcal{O}_F} \to \mathrm{Tate}_{\mathfrak{c}\mathfrak{q},\mathfrak{q}^{-1}}$$

[1]We are supposing here for simplicity that the element $\mathfrak{c}\mathfrak{q}^2$ belongs to our fixed set or representatives $\mathfrak{C}$, since the induced action of $S_{\mathfrak{q}}^{\vee,\circ}$ is independent of this choice.

which induces, via the canonical identification (2.13), the following diagram

$$\begin{tikzcd}
\omega^{k,\ell}_{\mathrm{Tate}_{\mathfrak{c},\mathcal{O}_F}/S_X} \arrow[d, equal, "\mathrm{can}(\mathfrak{c},\mathcal{O}_F)"'] & & \omega^{k,\ell}_{\mathrm{Tate}_{\mathfrak{c}\mathfrak{q},\mathfrak{q}^{-1}}/S_X} \arrow[ll, "\dot{S}_{\mathfrak{q}}"'] \arrow[d, equal, "\mathrm{can}(\mathfrak{c}\mathfrak{q},\mathfrak{q}^{-1})"] \\
(\mathfrak{c}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_X} & & (\mathfrak{c}\mathfrak{q}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}\mathfrak{q}^2\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_X}, \arrow[ll, "\dot{S}_{\mathfrak{q}}"']
\end{tikzcd} \tag{3.6}$$

where the bottom map is induced by the natural inclusions $\mathfrak{c}\mathfrak{q} \subset \mathfrak{c}$ and $\mathfrak{c}\mathfrak{q}^2 \subset \mathfrak{c}$. It is clear that in the case of $\mathfrak{q} = \mathfrak{p}$, a prime above p, this natural inclusion would not induce an isomorphism over $\mathcal{O}$. In the case of a paritious weight (k,ℓ), i.e. such that $k_\tau + 2\ell_\tau = \mathsf{w} \in \mathbb{Z}$ for all $\tau \in \Sigma$, the introduced normalization is essential to make the diamond operator invertible on the sheaf of paritious Hilbert modular forms over $\mathcal{O}$ for prime ideals of $\mathcal{O}_F$ co-prime with $\mathfrak{n}$. For non-paritious weights, it is not clear how to make the geometric operator $\dot{S}_{\mathfrak{p}}$ invertible, for places $\mathfrak{p}$ dividing p. Let us look at the map

$$(\mathfrak{c}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_X} \xleftarrow{(\mathrm{Nm}(\mathfrak{q}))^{-\mathsf{w}}\dot{S}_{\mathfrak{q}}} (\mathfrak{c}\mathfrak{q}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}\mathfrak{q}^2\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_X}.$$

We recall that $(\mathfrak{c}\otimes\mathcal{O})^k = \bigotimes_{\tau\in\Sigma}(\mathfrak{c}\otimes\mathcal{O})_\tau^{\otimes k_\tau}$ is an $\mathcal{O}$-module of rank 1 (see Remark 2.3.4). In particular, each $(\mathfrak{c}\otimes\mathcal{O})_\tau$ is a principal ideal in $\mathcal{O}$. So we consider the following morphism of $\mathcal{O}$-modules:

$$\begin{aligned}
\bigotimes_{\tau\in\Sigma}(\mathfrak{c}\otimes\mathcal{O})_\tau = (\mathfrak{c}\otimes\mathcal{O})^{\mathsf{t}} &\longleftarrow (\mathfrak{c}\mathfrak{q}\otimes\mathcal{O})^{\mathsf{t}} = \bigotimes_{\tau\in\Sigma}(\mathfrak{c}\mathfrak{q}\otimes\mathcal{O})_\tau \\
\mathrm{Nm}(\mathfrak{q})^{-1}(a_{\tau_1}\otimes\ldots\otimes a_{\tau_d}) &\longleftarrow\!\shortmid (a_{\tau_1}\otimes\ldots\otimes a_{\tau_d})
\end{aligned}$$

, where $\mathsf{t} \in \mathbb{Z}^\Sigma$ denotes the weight vector with 1 in each entry. This is in particular an isomorphism for any prime ideal $\mathfrak{q}$ of $\mathcal{O}_F$. In fact, for $\mathfrak{q}$ not dividing p, this is clearly an isomorphism, since $\mathrm{Nm}(\mathfrak{q}) \in \mathcal{O}^\times$. For $\mathfrak{q} = \mathfrak{p}$ a prime above p, an element $a \in (\mathfrak{c}\mathfrak{p}\otimes\mathcal{O})^{\mathsf{t}} \simeq \mathrm{Nm}(\mathfrak{p})\mathcal{O}$ has $\mathfrak{p}$-valuation $v_{\mathfrak{p}}(a) \geq 1$, so the element $\mathrm{Nm}(\mathfrak{p})^{-1}$ has $\mathfrak{p}$-valuation greater or equal to 0, and it belongs to $(\mathfrak{c}\otimes\mathcal{O})^{\mathsf{t}} \simeq \mathcal{O}$. Since $k_\tau + 2\ell_\tau = \mathsf{w} \in \mathbb{Z}$, the map

$$\begin{aligned}
(\mathfrak{c}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_X} &\longleftarrow (\mathfrak{c}\mathfrak{q}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}\mathfrak{q}^2\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell \otimes_{\mathcal{O}} \mathcal{O}_{S_X} \\
\sum_{\xi\in X_+}(\mathrm{Nm}(\mathfrak{q})^{-\mathsf{w}}a_\xi)q^\xi &\longleftarrow\!\shortmid \sum_{\xi\in X_+} a_\xi q^\xi
\end{aligned}$$

is an isomorphism of $\mathcal{O}$-modules. This extends to any $\mathcal{O}$-module R.

For primes $\mathfrak{p} \subset \mathcal{O}_F$ above p, one can construct normalized diamond operators also for non-paritious weights, using the uniformizer $\varpi_{\mathfrak{p}}$. In particular, by applying the same reasoning as above to the normalizing factor $\prod_{\tau\in\Sigma_{\mathfrak{p}}}\tau(\varpi_{\mathfrak{p}})^{-(k_\tau+2\ell_\tau)}$ instead of $\mathrm{Nm}(\mathfrak{p})^{-\mathsf{w}}$, one gets an equivalent of Equation 3.5 :

$$S_{\mathfrak{p}}^{\vee,\circ} : \mathrm{H}^0(\mathrm{Sh}^{\mathrm{tor}},\omega^{k,\ell}) \xrightarrow[\sim]{\prod_{\tau\in\Sigma_{\mathfrak{p}}}\tau(\varpi_{\mathfrak{p}})^{-(k_\tau+2\ell_\tau)}S_{\mathfrak{p}}^\vee} \mathrm{H}^0(\mathrm{Sh}^{\mathrm{tor}},\omega^{k,\ell}). \tag{3.7}$$

Definition 3.1.2. Let $(k,\ell) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$ and let R be any $\mathcal{O}$-algebra satisfying Hypothesis 5. Let $\mathfrak{p} \subset \mathcal{O}_F$ be a prime dividing p. We define the *normalized diamond operator at* $\mathfrak{p}$

$$\langle\mathfrak{p}\rangle_{k,\ell} : \mathrm{H}^0(\mathrm{Sh}^{\mathrm{tor}}_R,\dot{\omega}^{k,\ell}) \to \mathrm{H}^0(\mathrm{Sh}^{\mathrm{tor}}_R,\dot{\omega}^{k,\ell}_R)$$

to be $\langle\mathfrak{p}\rangle_{k,\ell} := \left(\prod_{\tau\in\Sigma_{\mathfrak{p}}}\tau(\varpi_{\mathfrak{p}})^{-(k_\tau+2\ell_\tau)}S_{\mathfrak{p}}^\vee\right)^{-1}$, the inverse of the properly normalized isomorphism of Equation (3.7).

3.2 Hilbert Moduli Space with Iwahori level structure

In order to construct Hecke operators at a prime $\mathfrak{p}$ dividing p, one has to look at Hilbert modular schemes with extra $\Gamma_0(\mathfrak{p})$- structure, which here will be taken to be the Iwahori level structures as in [ERX17a, Section 3.1], first constructed by Pappas ([Pap95]) and Pappas-Rapoport ([PR05]). Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_F$ dividing p, f its residual degree and e its inertia degree. For a chosen representative $\mathfrak{c} \in \mathfrak{C}$, let $\alpha \in F^\times$ be such that $\mathfrak{c}\mathfrak{p} = \alpha\mathfrak{c}'$, for $\mathfrak{c}' \in \mathfrak{C}$ another representative in Cl_F^+. The following definition is taken as in [ERX17a, Section 3.1].

Definition 3.2.1. Let $\mathcal{M}_\mathfrak{c}(\mathfrak{n};\mathfrak{p})$ denote the functor associating to a locally noetherian $\mathcal{O}$-scheme S the set of isomorphism classes of tuples $\big((A,\lambda,\mu,\mathcal{F});(A',\lambda',\mu',\mathcal{F}');\phi;\psi\big)$, where

- $(A,\lambda,\mu,\mathcal{F})$ is an S-point of $\mathrm{Y}_\mathfrak{c}$;
- $(A',\lambda',\mu',\mathcal{F}')$ is an S-point of $\mathrm{Y}_{\mathfrak{c}'}$;
- $\phi : A \to A'$ and $\psi : A' \to A \otimes \mathfrak{c}(\mathfrak{c}')^{-1}$ are $\mathcal{O}_F$-equivariant S-isogenies such that:
  - $\deg(\phi) = p^f = \deg(\psi)$;
  - the compositions $\psi \circ \phi$ and $(\phi \otimes \mathfrak{c}(\mathfrak{c}')^{-1}) \circ \psi$ are the natural isogenies $A \to A \otimes \mathfrak{c}(\mathfrak{c}')^{-1}$ and $A' \to A' \otimes \mathfrak{c}(\mathfrak{c}')^{-1}$ induced by $\mathcal{O}_F \subseteq \mathfrak{p}^{-1} \overset{\cdot\alpha}{\simeq} \mathfrak{c}(\mathfrak{c}')^{-1}$;
  - ϕ is compatible with polarizations, i.e. $\phi \circ \lambda \circ \phi^\vee = \tilde{\lambda}'$, where $\tilde{\lambda}' : (A')^\vee \to A' \otimes \mathfrak{c}$ is the map induced by composing λ' with $\mathfrak{c}' \overset{\cdot\alpha}{\simeq} \mathfrak{c}\mathfrak{p} \subset \mathfrak{c}$;
  - ϕ and ψ are compatible with level structures, i.e. $\phi \circ \mu = \mu'$ and $\psi \circ \mu' = \mu \otimes \mathfrak{c}(\mathfrak{c}')^{-1}$;
  - ϕ and ψ are compatible with the filtrations, i.e. for any $\mathfrak{p}'$ dividing p, and for any $j \in \{1, \dots f_{\mathfrak{p}'}\}$ the morphisms of S-modules
  $$\phi^* : \omega_{A'/S,\mathfrak{p}',j} \to \omega_{A/S,\mathfrak{p}',j} \text{ and } \psi^* : \omega_{A/S,\mathfrak{p}',j} \simeq \omega_{A/S,\mathfrak{p}',j} \otimes \mathfrak{c}(\mathfrak{c}')^{-1} \to \omega_{A'/S,\mathfrak{p}',j},$$
  preserve the filtrations $\mathcal{F}^\bullet_{\mathfrak{p}',j}$ and $\mathcal{F}'^\bullet_{\mathfrak{p}',j}$.

This functor is representable by an $\mathcal{O}$-scheme of finite type that we will denote $\mathrm{Y}_\mathfrak{c}(\mathfrak{p})$ ([ERX17a, Section 3.1]).

There are two natural forgetful maps:

$$\begin{tikzcd} & \mathrm{Y}_\mathfrak{c}(\mathfrak{p}) \arrow[dl, "\pi_{1,\alpha}"'] \arrow[dr, "\pi_{2,\alpha}"] & \\ \mathrm{Y}_\mathfrak{c} & & \mathrm{Y}_{\mathfrak{c}'} \end{tikzcd} \tag{3.8}$$

induced by keeping only the appropriate data of HBAV, i.e.

$$\begin{tikzcd} & \big((A,\lambda,\mu,\mathcal{F});(A',\lambda',\mu',\mathcal{F}');\phi;\psi\big) \arrow[dl, mapsto, "\pi_{1,\alpha}"'] \arrow[dr, mapsto, "\pi_{2,\alpha}"] & \\ (A,\lambda,\mu,\mathcal{F}) & & (A',\lambda',\mu',\mathcal{F}')\,. \end{tikzcd}$$

As seen for $\mathrm{Y}_{\mathfrak{c}}$ in Section **??** the group E acts freely on $\mathrm{Y}_{\mathfrak{c}}(\mathfrak{p})$, by acting at the same time on A and A', and hence we denote by $\mathrm{Sh}_{\mathfrak{c}}(\mathfrak{p})$ the corresponding quotient. As before, we set

$$\mathrm{Y}(\mathfrak{p}) = \coprod_{\mathfrak{c}} \mathrm{Y}_{\mathfrak{c}}(\mathfrak{p}) \ , \qquad \mathrm{Sh}(\mathfrak{p}) = \coprod_{\mathfrak{c}} \mathrm{Sh}_{\mathfrak{c}}(\mathfrak{p}) \ .$$

Since $\pi_{1,\alpha}, \pi_{2,\alpha}$ are both equivariant under the action of E, we have induced projections:

$$\begin{tikzcd} & \mathrm{Sh}(\mathfrak{p}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathrm{Sh} & & \mathrm{Sh}\ , \end{tikzcd}$$

which are independent of the choices of α by [ERX17a, Equation 3.1.2]. Moreover, by [ERX17a, Proposition 3.7], these morphisms of $\mathcal{O}$-schemes are finite and flat over the ordinary locus of $\mathrm{Sh}(\mathfrak{p})$. One can construct smooth toroidal compactifications for the splitting models with Iwahori level structures as in [RX17, Section 2.11] and extend the above maps π_1, π_2 to maps $\mathrm{Sh}(\mathfrak{p})^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}}$, as in [ERX17a, Section 3.9], which may no longer be finite and flat over the ordinary locus.

3.3 Hecke Operator at $\mathfrak{p}$ over $\mathcal{O}$

We first recall the definition of the normalized Hecke operator at $\mathfrak{p}$ as given by Emerton, Reduzzi and Xiao. In their construction of the Hecke operator at $\mathfrak{p}$, Emerton, Reduzzi and Xiao have to suppose the following for the weight $k \in \mathbb{Z}^{\Sigma}$ (see [ERX17a, Conditions 3.11.1]).

Hypothesis 6. Assume that the weights k_τ for $\tau \in \Sigma_{\mathfrak{p}}$ satisfy the following:

- $\sum_{\tau \in \Sigma_{\mathfrak{p}}} k_\tau \geq ef$;
- $k_{\tau_{\mathfrak{p},i}^{(j+1)}} \geq k_{\tau_{\mathfrak{p},i}^{(j)}}$ for all $i = 1, \dots, f$ and $j = 1, \dots, e-1$;
- $pk_{\tau_{\mathfrak{p},i}^{(1)}} \geq k_{\tau_{\mathfrak{p},i}^{(e)}}$.

We would like to remark that by these conditions, one has that $k_\tau \geq 0$ for all $\tau \in \Sigma_{\mathfrak{p}}$. Moreover, these conditions correspond to what Diamond-Kassaei define as *minimal cone*, in [DK17] for unramified p and in [DK20] for general p, which we recall here.

Definition 3.3.1 (Diamond-Kassaei)**.** We say that a weight $k \in \mathbb{Z}^{\Sigma}$ belongs to the *minimal cone*, denoted C^{min}, if for every $\mathfrak{p}|p$:

- $k_{\tau_{\mathfrak{p},j}^{(i+1)}} \geq k_{\tau_{\mathfrak{p},j}^{(i)}}$ for all $j = 1, \dots, f_{\mathfrak{p}}$ and $i = 1, \dots, e_{\mathfrak{p}} - 1$;
- $pk_{\tau_{\mathfrak{p},j}^{(1)}} \geq k_{\tau_{\mathfrak{p},j-1}^{(e_{\mathfrak{p}})}}$ for all $j = 1, \dots f_{\mathfrak{p}}$.

We would like to stress that, in order to have a good theory of Hecke operators, one has to work with minimal weights. In particular, in Chapter 4 we will mainly work with weights that live in the minimal cone, or we will bring our forms to weights living in the minimal cone.

Definition 3.3.2 (Definition 3.12 of [ERX17a]). Let (k, ℓ) be a paritious weight, i.e. $k_\tau + 2\ell_\tau = \mathsf{w} \in \mathbb{Z}$, satisfying Hypothesis 6. Let $R_m := \mathcal{O}/\varpi^m\mathcal{O}$. The action of the *Hecke operator* $T_{\mathfrak{p}}^{\vee,\circ}$ on the cohomology of $\omega_{R_m}^{k,\ell}$ is defined as the composition of the following maps:

$$\begin{aligned} \mathrm{H}^i(\mathrm{Sh}^{\mathrm{tor}}, \omega_{R_m}^{k,\ell}) &\xrightarrow{\pi_2^*} \mathrm{H}^i(\mathrm{Sh}(\mathfrak{p})^{\mathrm{tor}}, \pi_2^*\omega_{R_m}^{k,\ell}) \\ &\xrightarrow{\pi_{1,*}} \mathrm{H}^i(\mathrm{Sh}^{\mathrm{tor}}, R\pi_{1,*}\pi_2^*\omega_{R_m}^{k,\ell}) \xrightarrow{\eta_m} \mathrm{H}^i(\mathrm{Sh}^{\mathrm{tor}}, \omega_{R_m}^{k,\ell}), \end{aligned}$$

where $\eta_m : R\pi_{1,*}\pi_2^*\omega_{R_m}^{k,\ell} \to \omega_{R_m}^{k,\ell}$ is a normalized morphism constructed from the dualizing trace map (see Introduction, Section 3.10 and Proposition 3.11 of [ERX17a]).

We want to remark that we denote this geometric normalized Hecke operator with a dual to distinguish it from the "arithmetic" normalized Hecke operator at $\mathfrak{p}$, which we will denote $T_{\mathfrak{p}}^{\circ}$. The two are dual of each other, as in Equation (3.16). As said in the introduction of this chapter, we want to work with "arithmetic" Hecke operators because they are the good ones to consider when working with the Galois representations attached to Hilbert modular forms.

In what follows, we will go through the steps to construct the Hecke operator $T_{\mathfrak{p}}^{\vee,\circ}$, to calculate its action on q-expansion. We maintain the generality of partial weights (k, ℓ), because we believe that the operator defined by Emerton, Reduzzi and Xiao can be extended to non-paritious minimal weights as well.

In order to understand the effect of the $T_{\mathfrak{p}}$ operator on q-expansions, it suffices to look first at the varieties $\mathrm{Y}_{\mathfrak{c}}(\mathfrak{p})$ and their toroidal compactifications, and therefore at the projection maps $\pi_{1,\alpha}, \pi_{2,\alpha}$ defined in (3.8). We will later take into account the passage to quotient $\mathrm{Sh}_{\mathfrak{c}}$ and its compactification.

Since we will be working with cusps $\mathfrak{c}\mathfrak{p}$ and $\mathfrak{c}\mathfrak{p}^{-1}$, which are not in our fixed set of representatives $\mathfrak{C}$, we will take $\alpha, \beta \in F^\times$ such that

$$\mathfrak{c}\mathfrak{p} = \alpha\mathfrak{c}' \quad \text{and} \quad \mathfrak{c}\mathfrak{p}^{-1} = \beta\mathfrak{c}'' ,$$

where $\mathfrak{c}', \mathfrak{c}''$ are in the set of chosen representatives $\mathfrak{C}$, coprime with p.

Proposition 3.3.3. *Let* $\infty(\mathfrak{c})$ *be the cusp at infinity and let* $\mathrm{Tate}_{\mathfrak{c},\mathcal{O}_F}/S_{\mathfrak{c}}$ *be the associated Tate variety (see Section 2.3). Then the inverse image under* $\pi_{1,\alpha}$ *of* $\infty(\mathfrak{c})$ *consists of two cusps, which will be labeled* $\infty_{\mathfrak{c}}$ *and* $0_{\mathfrak{c}}$*, the ramified one. In particular the inverse image under* $\pi_{1,\alpha}$ *of* $\mathrm{Tate}_{\mathfrak{c},\mathcal{O}_F} \to S_{\mathfrak{c}}$ *consists of*

- *an* $S_{\mathfrak{c}}$*-point on* $\mathrm{Y}_{\mathfrak{c}}(\mathfrak{p})$*, with* $A = (\mathbb{G}_m \otimes \mathfrak{c}^*)/q(\mathcal{O}_F)$ *and* $A' = (\mathbb{G}_m \otimes (\mathfrak{c}\mathfrak{p})^*)/q(\mathcal{O}_F)$ *over* $S_{\mathfrak{c}}$*;*
- *an* $S_{\mathfrak{c}\mathfrak{p}^{-1}}$*-point on* $\mathrm{Y}_{\mathfrak{c}}(\mathfrak{p})$*, with* $A = (\mathbb{G}_m \otimes \mathfrak{c}^*)/q(\mathcal{O}_F)$ *and* $A' = (\mathbb{G}_m \otimes \mathfrak{c}^*)/q(\mathfrak{p}^{-1})$*, over* $S_{\mathfrak{c}\mathfrak{p}^{-1}}$*.*

Proof. This follows from the construction of the Iwahori level structure and from [DW18, Proposition 3.3]. □

To calculate the action of $T_{\mathfrak{p}}^{\vee,\circ}$ on q-expansions on a form with coefficients in $R_m = \mathcal{O}/\varpi^m\mathcal{O}$, we will work with the schemes S_X over which the Tate object for the cusp $\infty(\mathfrak{c})$ lives. In particular, the module of q-expansions $\mathcal{M}_\infty^{k,\ell}(\mathfrak{c}; R_m)$ can be injected in a completed ring $R_X^\wedge \otimes_{\mathcal{O}} R_m$ (see the proof of Theorem 3.3.4), whose elements can be lifted in $\mathcal{O}$. We will then follow the steps of the construction of $T_{\mathfrak{p}}^{\vee,\circ}$ over $\mathcal{O}$ and we will reduce modulo ϖ^m the obtain equation. This can be done because the operator $T_{\mathfrak{p}}^{\vee,\circ}$ exists and is integral over $\mathcal{O}$.

Following the definition of the maps π_1, π_2, and by the above proposition, one gets the following diagram corresponding to the cusp $\infty_{\mathfrak{c}}$:

$$\begin{tikzcd}
& (\mathbb{G}_m \otimes \mathfrak{c}^*)/q(\mathcal{O}_F) \arrow[rr, "\phi"] \arrow[dr] \arrow[ddl, mapsto, "\pi_1"'] & & (\mathbb{G}_m \otimes (\mathfrak{c}\mathfrak{p})^*)/q(\mathcal{O}_F) \arrow[dl] \arrow[ddr, mapsto, "\pi_2"] & \\
& & S_{\mathfrak{c}} & & \\
(\mathbb{G}_m \otimes \mathfrak{c}^*)/q(\mathcal{O}_F) \arrow[d] & & & & (\mathbb{G}_m \otimes (\mathfrak{c}')^*)/q(\mathcal{O}_F) \arrow[d] \\
S_{\mathfrak{c}} & & & & S_{\mathfrak{c}'}
\end{tikzcd} \tag{3.9}$$

and the following diagram for the ramified cusp $0_{\mathfrak{c}}$:

$$\begin{tikzcd}
& (\mathbb{G}_m \otimes \mathfrak{c}^*)/q(\mathcal{O}_F) \arrow[rr, "\phi"] \arrow[dr] \arrow[ddl, mapsto, "\pi_1"'] & & (\mathbb{G}_m \otimes \mathfrak{c}^*)/q(\mathfrak{p}^{-1}) \arrow[dl] \arrow[ddr, mapsto, "\pi_2"] & \\
& & S_{\mathfrak{c}\mathfrak{p}^{-1}} & & \\
(\mathbb{G}_m \otimes \mathfrak{c}^*)/q(\mathcal{O}_F) \arrow[d] & & & & (\mathbb{G}_m \otimes \mathfrak{c}^*)/q(\mathfrak{p}^{-1}) \arrow[d] \\
S_{\mathfrak{c}} & & & & S_{\mathfrak{c}\mathfrak{p}^{-1}}
\end{tikzcd} \tag{3.10}$$

Since we want to work only with the standard cusps at infinity, i.e. those cusps labeled $\infty(\mathfrak{c})$ for $\mathfrak{c} \in \mathfrak{C}$, here we will use the natural diamond operator $S_{\mathfrak{q}}^{\vee}$ and the natural morpshim $\varphi_{\mathfrak{q}}$ from Section 3.1 to bring the cusp $\infty(\mathfrak{c}\mathfrak{p}, \mathfrak{p}^{-1})$, with corresponding Tate variety $(\mathbb{G}_m \otimes \mathfrak{c}^*)/q(\mathfrak{p}^{-1})/S_{\mathfrak{c}\mathfrak{p}^{-1}}$ to the cusp $\infty(\mathfrak{c}'')$. In this step, since we have fixed $\beta \in F_+$ such that $\beta\mathfrak{c}'' = \mathfrak{c}\mathfrak{p}^{-1}$, we will see the term β appear in the q-expansions.

3.3.1 Action on geometric q-expansions

We now have all the ingredients to prove the following

Theorem 3.3.4. *Let $R_m := \mathcal{O}/\varpi^m\mathcal{O}$ and let $(k, \ell) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$ satisfying Hypothesis 5 and Hypothesis 6. Let $f \in \mathrm{H}^0(\mathrm{Sh}_{R_m}, \omega_{R_m}^{k,\ell})$ and let $f = (f_{\mathfrak{c}})_{\mathfrak{c}\in\mathfrak{C}}$, where $f_{\mathfrak{c}} = \sum_{\xi\in\mathfrak{c}_+\cup\{0\}} a_\xi q^\xi$ be its geometric q-expansions at the cusp $\infty(\mathfrak{c})$. For a place $\mathfrak{p}$ of F above p, let $\alpha, \beta \in F_+$ be such that $\mathfrak{c}\mathfrak{p} = \alpha\mathfrak{c}'$ and $\mathfrak{c}\mathfrak{p}^{-1} = \beta\mathfrak{c}''$, for $\mathfrak{c}, \mathfrak{c}', \mathfrak{c}'' \in \mathfrak{C}$. Then for $\xi \in \mathfrak{c}_+$*

$$\begin{aligned} a_\xi\big((T_{\mathfrak{p}}^{\vee,\circ} f)_{\mathfrak{c}}\big) = \mathrm{Nm}(\mathfrak{p})^{-1} &\left(\prod_{\tau\in\Sigma_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{-\ell_\tau}\right) \alpha^{k+\ell} a_{\alpha^{-1}\xi}(f_{\mathfrak{c}'}) \\ &+ \left(\prod_{\tau\in\Sigma_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{k_\tau+\ell_\tau}\right) \beta^{k+\ell} a_{\beta^{-1}\xi}\big((S_{\mathfrak{p}}^{\vee,\circ} f)_{\mathfrak{c}''}\big), \end{aligned} \tag{3.11}$$

with $a_{\alpha^{-1}\xi} = 0$ if $\alpha^{-1}\xi \notin \mathfrak{c}'$ and $S_{\mathfrak{p}}^{\vee,\circ}$ is given in Equation 3.7. We recall that we denote by α^k the element $\prod_{\tau\in\Sigma} \tau(\alpha)^{k_\tau}$.

Remark 3.3.5. As we will see in the proof of Theorem 3.3.4, the formula in Equation 3.11 makes sense as it is over $\mathcal{O}$ and in particular it is integral over $\mathcal{O}$. Let us explain why. First of all, the coefficients $a_{\alpha^{-1}\xi}(f_{\mathfrak{c}'})$ and $a_{\beta^{-1}\xi}\big((S_{\mathfrak{p}}^{\vee,\circ} f)_{\mathfrak{c}'}\big)$ live respectively in the rank one $\mathcal{O}$-module $(\mathfrak{c}' \otimes \mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}'\mathfrak{d}^{-1} \otimes \mathcal{O})^\ell$ and $(\mathfrak{c}'' \otimes \mathcal{O})^k \otimes_{\mathcal{O}} (\mathfrak{c}''\mathfrak{d}^{-1} \otimes \mathcal{O})^\ell$, so they have non-negative ϖ-adic valuation. Let us remark that the normalization of the operator $S_{\mathfrak{p}}^{\vee,\circ}$ is again essential for the term $a_{\beta^{-1}\xi}\big((S_{\mathfrak{p}}^{\vee,\circ} f)_{\mathfrak{c}'}\big)$ to have non-negative $\mathfrak{p}$-valuation. In fact, thanks to the normalization, one has an isomorphism of rank one $\mathcal{O}$-modules $(\mathfrak{c}''\mathfrak{p}\otimes\mathcal{O})^k\otimes_{\mathcal{O}}(\mathfrak{c}''\mathfrak{p}^2\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell \to (\mathfrak{c}''\otimes\mathcal{O})^k\otimes_{\mathcal{O}}(\mathfrak{c}''\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell$. Now let us proceed to calculate the $\mathfrak{p}$-adic valuation of each addend of Equation (3.11). Since $\alpha\mathfrak{c}' = \mathfrak{c}\mathfrak{p}$ and $\mathfrak{c}, \mathfrak{c}'$ are coprime with p, $v_{\mathfrak{p}}(\alpha) = 1$. Therefore the $\mathfrak{p}$-adic valuation of the first term is

$$\begin{aligned} v_{\mathfrak{p}}(\text{first term}) &\geq -ef - \sum_{\tau\in\Sigma_{\mathfrak{p}}} \ell_\tau + \sum_{\tau\in\Sigma_{\mathfrak{p}}} (k_\tau + \ell_\tau) \\ &= \sum_{\tau\in\Sigma_{\mathfrak{p}}} k_\tau - ef \geq 0. \end{aligned}$$

The last equality is given by the first condition of Hypothesis 6.
Since $\beta\mathfrak{c}'' = \mathfrak{c}\mathfrak{p}^{-1}$ and $\mathfrak{c}, \mathfrak{c}''$ are coprime with p, $v_{\mathfrak{p}}(\beta) = -1$. Therefore the $\mathfrak{p}$-adic valuation of the second term is

$$v_{\mathfrak{p}}(\text{second term}) \geq \sum_{\tau\in\Sigma_{\mathfrak{p}}} k_\tau + \ell_\tau - \sum_{\tau\in\Sigma_{\mathfrak{p}}} (k_\tau + \ell_\tau) = 0$$

Therefore, the above Equation (3.11) taken over $\mathcal{O}$ is integral.

Proof. Let us give an argument to why we can work over $\mathcal{O}$ and then reduce modulo ϖ^m. By the construction of the toroidal compactification by Dimitrov ([Dim04, Théorème 7.2]), we can work over the schemes S_X at the chosen cusps. This is because the schemes S_X are by construction such that one has an open immersion $S_X \hookrightarrow \mathrm{Sh}_{\mathfrak{c}}^{\mathrm{tor}}$ and by formal completion one has also a morphism of schemes $S_X^\wedge \to S_X$, where $S_X^\wedge = \mathrm{Spf}(R_X^\wedge)$ denotes the formal completion of S_X. This induces for any $\mathcal{O}$-algebra R the following commutative diagram:

$$\begin{tikzcd} \mathrm{H}^0(\mathrm{Sh}_{\mathfrak{c},R}, \omega_R^{k,\ell}) \arrow[r] \arrow[d, hook] & \mathrm{H}^0(S_X \times \mathrm{Spec}(R), \omega_R^{k,\ell}) \arrow[r] & \mathrm{H}^0(S_X^\wedge \times \mathrm{Spec}(R), \omega_R^{k,\ell}) \arrow[d, hook] \\ \mathcal{M}_\infty^{k,\ell}(\mathfrak{c}; R) \arrow[rr] & & R_X^\wedge \otimes_{\mathcal{O}} R \end{tikzcd}$$

In particular, elements of $R_X^\wedge \otimes_{\mathcal{O}} R_m$ lift in characteristic 0 to $R_X^\wedge$. One has an action of the Hecke operators $T_{\mathfrak{p}}^{\vee,\circ}$ on the cusps (see for example Equation 3.9 and 3.10) and therefore over $R_X^\wedge \otimes_{\mathcal{O}} R_m$. In particular, one has the following commutative diagram:

$$\begin{tikzcd} \mathrm{H}^0(\mathrm{Sh}_{\mathfrak{c},R_m}, \omega_{R_m}^{k,\ell}) \arrow[r, hook] \arrow[d, "T_{\mathfrak{p}}^{\vee,\circ}"] & \mathcal{M}_\infty^{k,\ell}(\mathfrak{c}; R_m) \arrow[r, hook] \arrow[d, "T_{\mathfrak{p}}^{\vee,\circ}"] & R_X^\wedge \otimes_{\mathcal{O}} R_m \arrow[d, "T_{\mathfrak{p}}^{\vee,\circ}"] \\ \mathrm{H}^0(\mathrm{Sh}_{\mathfrak{c},R_m}, \omega_{R_m}^{k,\ell}) \arrow[r, hook] & \mathcal{M}_\infty^{k,\ell}(\mathfrak{c}; R_m) \arrow[r, hook] & R_X^\wedge \otimes_{\mathcal{O}} R_m \end{tikzcd}$$

where here we use red arrows to stress the fact that we are working over R_m, and that the vertical maps correspond to the Hecke operator $T_{\mathfrak{p}}^{\vee,\circ}$ acting on $\mathrm{H}^0(\mathrm{Sh}_{\mathfrak{c},R_m}, \omega_{R_m}^{k,\ell})$.
Moreover, over $\mathcal{O}$, one has a normalized Hecke operator $T_{\mathfrak{p}}^{\vee,\circ}$ on $\mathrm{H}^0(\mathrm{Sh}, \omega^{k,\ell})$ which is defined, as in Definition 3.3.2, as the composition $(\eta \circ \pi_{1,*} \circ \pi_2^*)$. We point out that η_m in Definition 3.3.2 of $T_{\mathfrak{p}}^{\vee,\circ}$ over $\mathrm{H}^0(\mathrm{Sh}_{R_m}, \omega_{R_m}^{k,\ell})$ is induced by the map η (see discussion before [ERX17a, Definition 3.12]). One has then the following commutative diagram:

$$\begin{tikzcd}
R_X^\wedge & \mathcal{M}_\infty^{k,\ell}(\mathfrak{c}) \arrow[l] \arrow[d, "(\eta\circ\pi_{1,*}\circ\pi_2^*)"] & \mathrm{H}^0(\mathrm{Sh}_{\mathfrak{c}}, \omega^{k,\ell}) \arrow[l] \arrow[d, "(\eta\circ\pi_{1,*}\circ\pi_2^*)"] \\
R_X^\wedge \arrow[from=u, "(\eta\circ\pi_{1,*}\circ\pi_2^*)"] & \mathcal{M}_\infty^{k,\ell}(\mathfrak{c}) \arrow[l] & \mathrm{H}^0(\mathrm{Sh}_{\mathfrak{c}}, \omega^{k,\ell}) \arrow[l]
\end{tikzcd}$$

where here we use blue arrows to stress the fact that the vertical arrows are in characteristic 0, corresponding to the Hecke operator $T_{\mathfrak{p}}^{\vee,\circ} = (\eta \circ \pi_{1,*} \circ \pi_2^*)$ on $\mathrm{H}^0(\mathrm{Sh}, \omega^{k,\ell})$. By construction, the map $R_X^\wedge \xrightarrow{(\eta\circ\pi_{1,*}\circ\pi_2^*)} R_X^\wedge$ reduces modulo ϖ^m to $R_X^\wedge \otimes_{\mathcal{O}} R_m \xrightarrow{T_{\mathfrak{p}}^{\vee,\circ}} R_X^\wedge \otimes_{\mathcal{O}} R_m$. Putting everything together, one has the following commutative diagram:

$$\begin{tikzcd}
 & & & R_X^\wedge \arrow[dl, two heads] \arrow[dd, "(\eta\circ\pi_{1,*}\circ\pi_2^*)"] & \mathcal{M}_\infty^{k,\ell}(\mathfrak{c}) \arrow[l, hook'] \arrow[dd, "(\eta\circ\pi_{1,*}\circ\pi_2^*)"] & \mathrm{H}^0(\mathrm{Sh}_{\mathfrak{c}}, \omega^{k,\ell}) \arrow[l, hook'] \arrow[dd, "(\eta\circ\pi_{1,*}\circ\pi_2^*)"] \\
\mathrm{H}^0(\mathrm{Sh}_{\mathfrak{c},R_m}, \omega_{R_m}^{k,\ell}) \arrow[r, hook] \arrow[dd, "T_{\mathfrak{p}}^{\vee,\circ}"] & \mathcal{M}_\infty^{k,\ell}(\mathfrak{c}; R_m) \arrow[r, hook] \arrow[dd, "T_{\mathfrak{p}}^{\vee,\circ}"] & R_X^\wedge \otimes_{\mathcal{O}} R_m \arrow[dd, "T_{\mathfrak{p}}^{\vee,\circ}"] & & & \\
 & & & R_X^\wedge \arrow[dl, two heads] & \mathcal{M}_\infty^{k,\ell}(\mathfrak{c}) \arrow[l, hook'] & \mathrm{H}^0(\mathrm{Sh}_{\mathfrak{c}}, \omega^{k,\ell}) \arrow[l, hook'] \\
\mathrm{H}^0(\mathrm{Sh}_{\mathfrak{c},R_m}, \omega_{R_m}^{k,\ell}) \arrow[r, hook] & \mathcal{M}_\infty^{k,\ell}(\mathfrak{c}; R_m) \arrow[r, hook] & R_X^\wedge \otimes_{\mathcal{O}} R_m & & &
\end{tikzcd}$$

One can therefore look at the action of $T_{\mathfrak{p}}^{\vee,\circ}$ on the $\mathcal{O}$-module $R_X^\wedge$ and then reduce modulo ϖ^m. The existence of compatible operators on $\mathrm{H}^0(\mathrm{Sh}_{\mathfrak{c},R_m}, \omega_{R_m}^{k,\ell})$ and $\mathrm{H}^0(\mathrm{Sh}_{\mathfrak{c},R}, \omega_R^{k,\ell})$ by construction of Emerton, Reduzzi and Xiao (see Definition 3.3.2) and the injectivity of the q-expansion maps assure that the obtained result is the image under the operators $T_{\mathfrak{p}}^{\vee,\circ}$ of the original modulo ϖ^m modular form. Here, we will compute the action of $T_{\mathfrak{p}}^{\vee,\circ}$ on the $\mathcal{O}$-modules $\mathcal{M}_\infty^{k,\ell}(\mathfrak{c})$ by doing $\mathcal{O}$-integral steps between these modules. We will at the end reduce the obtained equation modulo ϖ^m.

We now follow diagrams (3.9) and (3.10) to compute the corresponding effect of q-expansions. It will then suffice to add the results to obtained the desired equation.

Let us start by the cusp $\infty_{\mathfrak{c}}$. Following the diagram (3.9), the sheaf $\omega^{k,\ell}$ can be trivialized as explained in Remark 2.3.4, yielding the following chain of homomorpshims of $\mathcal{O}$-modules (recall that the sheaves ω_τ and δ_τ are contra-variant):

$$\begin{tikzcd}
\mathcal{M}^{k,\ell}_\infty(\mathfrak{c}') \arrow[d, "\pi_2^*"] & \sum_{\xi'\in\mathfrak{c}'_+\cup\{0\}} a_{\xi'}q^{\xi'} \arrow[d, mapsto, "\text{Lemma 2.4.3}"] \\
\mathcal{M}^{k,\ell}_{\mathfrak{cp},\mathcal{O}_F}(\mathfrak{cp}) \arrow[d, hook, "\phi^*"] & \sum_{\xi\in(\mathfrak{cp})_+\cup\{0\}} \alpha^{k+\ell}a_{\alpha^{-1}\xi}q^{\xi} \arrow[d, mapsto] \\
\mathcal{M}^{k,\ell}_{\mathfrak{c},\mathcal{O}_F}(\mathfrak{c}) \arrow[d, "\mathrm{Nm}(\mathfrak{p})^{-1}(\eta\circ\pi_{1,*})"] & \alpha^{k+\ell}\sum_{\xi\in\mathfrak{c}_+\cup\{0\}} a_{\alpha^{-1}\xi}q^{\xi} \arrow[d, mapsto] \\
\mathcal{M}^{k,\ell}_\infty(\mathfrak{c}) & \mathrm{Nm}(\mathfrak{p})^{-1}\prod_{\tau\in\Sigma_\mathfrak{p}}\tau(\varpi_\mathfrak{p})^{-\ell_\tau}\alpha^{k+\ell}\sum_{\xi\in\mathfrak{c}_+\cup\{0\}} a_{\alpha^{-1}\xi}q^{\xi}
\end{tikzcd} \tag{3.12}$$

Let us recall that in the last step, the map η (see [ERX17a, Section 3.10]) is obtained via the dualizing trace map and it contains the normalization factor $\prod_{\tau\in\Sigma_\mathfrak{p}}\tau(\varpi_\mathfrak{p})^{-\ell_\tau}$. Moreover, the middle step corresponding to the isogeny ϕ is the natural inclusion given by Proposition 2.3.5.

Let us now look at the cusp $0_\mathfrak{c}$. We first have to complete the diagram (3.10) in order to start from a cusp at infinity, $\infty(\mathfrak{c}'')$. Recall that $\mathfrak{cp}^{-1}=\beta\mathfrak{c}''$, so we first re-elaborate the Tate object in order to write it over $S_{\mathfrak{c}''}$ and then we apply the map $\varphi_\mathfrak{q}$ (see Section 3.1).

$$\begin{tikzcd}
\mathbb{G}_m\otimes\mathfrak{c}^*/q(\mathfrak{p}^{-1}) \arrow[r, "1\otimes\beta"] \arrow[d] & \mathbb{G}_m\otimes(\mathfrak{c}''\mathfrak{p})^*/q(\mathfrak{p}^{-1}) \arrow[d] & \mathbb{G}_m\otimes(\mathfrak{c}'')^*/q(\mathcal{O}_F) \arrow[l, "\varphi_\mathfrak{p}"'] \arrow[dl] \\
S_{\mathfrak{cp}^{-1}} \arrow[r] & S_{\mathfrak{c}''} &
\end{tikzcd} \tag{3.13}$$

The last morphism is the one defining the operator $S_\mathfrak{p}^\vee$ identifying the q-expansion of a HMF f at $\infty(\mathfrak{c}''\mathfrak{p}^2,\mathfrak{p}^{-1})$ with the q-expansion of $\left(S_\mathfrak{p}^\vee f\right)_{\mathfrak{c}''}$ at $\infty(\mathfrak{c}'')$, which we will write as $\sum_{\xi''\in\mathfrak{c}''_+\cup\{0\}} b_{\xi''}q^{\xi''}$. In particular, since we haven't yet normalized the operators $S_\mathfrak{p}^\vee$, it is clear (see diagram (3.6)) that the elements $b_{\xi''}$ come from the rank one $\mathcal{O}$-module $(\mathfrak{c}''\mathfrak{p}\otimes\mathcal{O})^k\otimes_\mathcal{O}(\mathfrak{c}''\mathfrak{p}^2\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell\hookrightarrow(\mathfrak{c}''\otimes\mathcal{O})^k\otimes_\mathcal{O}(\mathfrak{c}''\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell$ and have $\mathfrak{p}$-valuation greater of equal than $\sum_{\tau\in\Sigma_\mathfrak{p}}(k_\tau+2\ell_\tau)$. The first square of diagram (3.13) induces, similarly to Lemma 2.4.3, a multiplication by $\beta^{k+\ell}$ on the sheaves. In fact, staring from the following commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_F \arrow[r] \arrow[d, equal] & \mathbb{G}_m\otimes\mathfrak{c}^* \arrow[r] \arrow[d, "1\otimes\beta"] & \mathbb{G}_m\otimes\mathfrak{c}^*/q(\mathfrak{p}^{-1}) \arrow[r] \arrow[d, "1\otimes\beta"] \arrow[dr] & 0 \\
 & & & & S_{\mathfrak{c}''} \\
0 \arrow[r] & \mathcal{O}_F \arrow[r] & \mathbb{G}_m\otimes(\mathfrak{c}''\mathfrak{p})^* \arrow[r] & \mathbb{G}_m\otimes(\mathfrak{c}''\mathfrak{p})^*/q(\mathfrak{p}^{-1}) \arrow[r] \arrow[ur] & 0
\end{tikzcd}$$

one obtains via the canonical identification 2.13a morphism on the $\mathcal{O}$-modules of coefficients

$$\begin{aligned}
(\mathfrak{c}''\mathfrak{p}\otimes\mathcal{O})^k\otimes(\mathfrak{c}''\mathfrak{p}^2\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell &\to (\mathfrak{c}\otimes\mathcal{O})^k\otimes(\mathfrak{cp}\mathfrak{d}^{-1}\otimes\mathcal{O})^\ell \\
b_{\xi''} &\mapsto \beta^{k+\ell}b_{\xi''}\ ,
\end{aligned}$$

which gives a morphism of $\mathcal{O}$-modules $\mathcal{M}^{k,\ell}_{\mathfrak{c}''\mathfrak{p},\mathfrak{p}^{-1}}(\mathfrak{c}'') \xrightarrow{\cdot\beta^{k+\ell}} \mathcal{M}^{k.\ell}_{\mathfrak{c},\mathfrak{p}^{-1}}(\mathfrak{c}\mathfrak{p}^{-1})$, mapping

$$\sum_{\xi''\in\mathfrak{c}''_+\cup\{0\}} b_{\xi''}q^{\xi''} \mapsto \sum_{\xi''\in\mathfrak{c}''_+\cup\{0\}} \beta^{k+\ell}b_{\xi''}q^{\beta\xi''} = \sum_{\xi\in(\mathfrak{c}\mathfrak{p}^{-1})_+\cup\{0\}} \beta^{k+\ell}b_{\beta^{-1}\xi}q^{\xi}.$$

Now we can go through the diagram (3.10) for the cusp $0_{\mathfrak{c}}$:

$$\begin{array}{ccc} \mathcal{M}^{k,\ell}_{\mathfrak{c},\mathfrak{p}^{-1}}(\mathfrak{c}\mathfrak{p}^{-1}) & & \sum_{\xi\in(\mathfrak{c}\mathfrak{p}^{-1})_+\cup\{0\}}\beta^{k+\ell}b_{\beta^{-1}\xi}q^{\xi} \\ \Big\| {\scriptstyle \pi_2^*} & & \downarrow \\ \mathcal{M}^{k,\ell}_{\mathfrak{c},\mathfrak{p}^{-1}}(\mathfrak{c}\mathfrak{p}^{-1}) & & \sum_{\xi\in(\mathfrak{c}\mathfrak{p}^{-1})_+\cup\{0\}}\beta^{k+\ell}b_{\beta^{-1}\xi}q^{\xi} \\ \big\downarrow {\scriptstyle \phi^*} & & \downarrow \\ \mathcal{M}^{k,\ell}_{\mathfrak{c},\mathcal{O}_F}(\mathfrak{c}\mathfrak{p}^{-1}) & & \beta^{k+\ell}\sum_{\xi\in(\mathfrak{c}\mathfrak{p}^{-1})_+\cup\{0\}}b_{\beta^{-1}\xi}q^{\xi} \\ \big\downarrow {\scriptstyle (\eta\circ\pi_{1,*})} & & \downarrow \\ \mathcal{M}^{k,\ell}_{\infty}(\mathfrak{c}) & & \prod_{\tau\in\Sigma_{\mathfrak{p}}}\tau(\varpi_{\mathfrak{p}})^{-\ell_\tau}\beta^{k+\ell}\sum_{\xi\in\mathfrak{c}_+\cup\{0\}}b_{\beta^{-1}\xi}q^{\xi}\ . \end{array} \tag{3.14}$$

We recall as above that in the last step the map η contains the normalization factor $\prod_{\tau\in\Sigma_{\mathfrak{p}}}\tau(\varpi_{\mathfrak{p}})^{-\ell_\tau}$. Moreover, the middle step corresponding to the isogeny ϕ is the natural inclusion given by Proposition 2.3.6. We know what to rewrite the obtained factor using the normalized operator $S^{\vee,circ}_{\mathfrak{p}}$ as given in Equation 3.7. Recall that $b_{\xi''} = a_{\xi''}\big((S^{\vee}_{\mathfrak{p}}f)_{\mathfrak{c}''}\big)$ and in particular, one has that $\prod_{\tau\in\Sigma_{\mathfrak{p}}}\tau(\varpi_{\mathfrak{p}})^{-(k_\tau+2\ell_\tau)}b_{\xi''} = a_{\xi''}\big((S^{\vee,\circ}_{\mathfrak{p}}f)_{\mathfrak{c}''}\big)$. Therefore, the last equation of the above diagram can be rewritten as

$$\prod_{\tau\in\Sigma_{\mathfrak{p}}}\tau(\varpi_{\mathfrak{p}})^{k_\tau+\ell_\tau}\beta^{k+\ell}\sum_{\xi\in\mathfrak{c}_+\cup\{0\}}a_{\xi}\big((S^{\vee,\circ}_{\mathfrak{p}}f)_{\mathfrak{c}''}\big). \tag{3.15}$$

Adding together the last equation of diagram 3.12 for the cusp $\infty_{\mathfrak{c}}$ and Equation 3.15 for the cusp $0_{\mathfrak{c}}$ gives the desired result.

□

We now assume that the weights $(k,\ell)\in\mathbb{Z}^{\Sigma}\times\mathbb{Z}^{\Sigma}$, satisfying Hypothesis 6, are paritious, i.e. $k_\tau+2\ell_\tau = \mathsf{w}\in\mathbb{Z}$ for all $\tau\in\Sigma$. We then use the normalized diamond operator $\langle\mathfrak{p}\rangle_{\mathsf{w}}$, as defined in Definition 3.1.1, to set the normalized Hecke operator at $\mathfrak{p}$ to be

$$T^{\circ}_{\mathfrak{p}} = T^{\vee,\circ}_{\mathfrak{p}}\circ\langle\mathfrak{p}\rangle_{\mathsf{w}}\ . \tag{3.16}$$

To lighten the formulas, we will denote by $\varpi_{\mathfrak{p}}^{\ell_{\mathfrak{p}}}$ the product $\prod_{\tau\in\Sigma_{\mathfrak{p}}}\tau(\varpi_{\mathfrak{p}})^{\ell_\tau}$.

Corollary 3.3.6. *Let $R_m := \mathcal{O}/\varpi^m\mathcal{O}$ and let $(k,\ell)\in\mathbb{Z}^{\Sigma}\times\mathbb{Z}^{\Sigma}$ be a paritious weight, i.e. $k_\tau+2\ell_\tau=\mathsf{w}$ for all $\tau\in\Sigma$. Suppose that R_m and the weights (k,ℓ) satisfy Hypothesis 5 and Hypothesis 6. Let $f\in\mathrm{H}^0(\mathrm{Sh}_{R_m},\omega^{k,\ell}_{R_m})$ and let $f=\big(f_{\mathfrak{c}}\big)_{\mathfrak{c}\in\mathrm{Cl}^+_F}$, where $f_{\mathfrak{c}}=\sum_{\xi\in\mathfrak{c}_+\cup\{0\}}a_\xi q^\xi$ is its geometric q-expansions at the cusp $\infty(\mathfrak{c})$. For a place $\mathfrak{p}$ of F above p, let $\alpha,\beta\in F_+$ be such that $\mathfrak{c}\mathfrak{p}=\alpha\mathfrak{c}'$ and $\mathfrak{c}\mathfrak{p}^{-1}=\beta\mathfrak{c}''$, for $\mathfrak{c},\mathfrak{c}',\mathfrak{c}''\in\mathrm{Cl}^+_F$. Then for $\xi\in\mathfrak{c}_+$*

$$\begin{aligned} a_\xi\big((T^{\circ}_{\mathfrak{p}}f)_{\mathfrak{c}}\big) =& \mathrm{Nm}(\mathfrak{p})^{\mathsf{w}-1}\left(\frac{\mathrm{Nm}(\mathfrak{c})}{\mathrm{Nm}(\mathfrak{c}')}\right)^{\mathsf{w}}\varpi_{\mathfrak{p}}^{-\ell_{\mathfrak{p}}}\alpha^{-\ell}a_{\alpha^{-1}\xi}\left(\big(\langle\mathfrak{p}\rangle_{\mathsf{w}}f\big)_{\mathfrak{c}'}\right) \\ &+\left(\frac{\mathrm{Nm}(\mathfrak{c})}{\mathrm{Nm}(\mathfrak{c}'')}\right)^{\mathsf{w}}\varpi_{\mathfrak{p}}^{-\ell_{\mathfrak{p}}}\beta^{-\ell}a_{\beta^{-1}\xi}(f_{\mathfrak{c}''}), \end{aligned} \tag{3.17}$$

with $a_{\alpha^{-1}\xi} = 0$ *if* $\alpha^{-1}\xi \notin \mathfrak{c}'$.

Let us remark that our geometric coefficients depend on the choice of fixed representatives of Cl_F^+, so we can normalize the geometric coefficients to get better readable formulas.

Definition 3.3.7. Let $(k,\ell) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$ be a paritous weight, i.e. $k_\tau + 2\ell_\tau = \mathsf{w}$ for all $\tau \in \Sigma$. Let $f \in \mathrm{H}^0(\mathrm{Sh}, \omega^{k,\ell})$ and let $f = \big(f_\mathfrak{c}\big)_{\mathfrak{c}\in Cl_F^+}$, where $f_\mathfrak{c} = \sum_{\xi\in\mathfrak{c}_+\cup\{0\}} a_\xi q^\xi$ be its geometric q-expansions at the cusp $\infty(\mathfrak{c})$. We define the *normalized geometric coefficients* as

$$a_\xi^\circ\big(f_\mathfrak{c}\big) := \mathrm{Nm}(\mathfrak{c})^{-\mathsf{w}} a_\xi\big(f_\mathfrak{c}\big) \ .$$

Remark 3.3.8. With this notation, one can re-write the above Equation (3.17) as

$$a_\xi^\circ\big((T_\mathfrak{p}^\circ f)_\mathfrak{c}\big) = \mathrm{Nm}(\mathfrak{p})^{\mathsf{w}-1} \varpi_\mathfrak{p}^{-\ell_\mathfrak{p}} \alpha^{-\ell} a_{\alpha^{-1}\xi}^\circ\Big(\big(\langle\mathfrak{p}\rangle_\mathsf{w} f\big)_{\mathfrak{c}'}\Big) + \varpi_\mathfrak{p}^{-\ell_\mathfrak{p}} \beta^{-\ell} a_{\beta^{-1}\xi}^\circ(f_{\mathfrak{c}''}) \ . \tag{3.18}$$

Now, recalling that $v_\mathfrak{p}(\alpha) = 1$ and $v_\mathfrak{p}(\beta) = -1$, it is clear that $v_\mathfrak{p}\left(a_\xi^\circ\big((T_\mathfrak{p} f)_\mathfrak{c}\big)\right) \geq 0$. In fact,

$$v_\mathfrak{p}\left(a_\xi^\circ\big((T_\mathfrak{p}^\circ f)_\mathfrak{c}\big)\right) \geq \min\left(\sum_{\tau\in\Sigma_\mathfrak{p}} (k_\tau + 2\ell_\tau) - ef - \sum_{\tau\in\Sigma_\mathfrak{p}} 2\ell_\tau, \sum_{\tau\in\Sigma_\mathfrak{p}} (-\ell_\tau + \ell_\tau)\right) = 0 \ .$$

So for any $\mathfrak{c} \in \mathfrak{C}$ and $\xi \in \mathfrak{c}_+$, $a_\xi\big((T_\mathfrak{p}^\circ f)_\mathfrak{c}\big)$ lies in $\mathcal{O}$, making it possible to consider this operator modulo ϖ. Moreover, it is not the 0 operator modulo ϖ.

Remark 3.3.9. We also would like to remark that the construction and the computations of Theorem 3.3.4 work also for a prime $\mathfrak{q} \subset \mathcal{O}_F$ not dividing p, with a non-normalized map η of Definition 3.3.2. Therefore, one obtains the action of the Hecke operator $T_\mathfrak{q}$:

$$a_\xi^\circ\big((T_\mathfrak{q} f)_\mathfrak{c}\big) = \mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1} \alpha^{-\ell} a_{\alpha^{-1}\xi}^\circ\left(\big(\langle\mathfrak{q}\rangle_\mathsf{w} f\big)_{\mathfrak{c}'}\right) + \beta^{-\ell} a_{\beta^{-1}\xi}^\circ(f_{\mathfrak{c}''}) \ , \tag{3.19}$$

where now $\alpha, \beta \in F_+$ and $\mathfrak{c}', \mathfrak{c}'' \in \mathfrak{C}$ are such that $\alpha\mathfrak{c}' = \mathfrak{c}\mathfrak{q}$ and $\beta\mathfrak{c}'' = \mathfrak{c}\mathfrak{q}^{-1}$.

Let us now proceed with the proof Corollary 3.3.6.

Proof. First of all, let us look at the last equation of diagram 3.14 and recall that $b_{\xi''} = a_{\xi''}\big((S_\mathfrak{p}^\vee f)_{\mathfrak{c}''}\big)$. Since $\beta \in F_+$ is such that $\beta\mathfrak{c}'' = \mathfrak{c}\mathfrak{p}^{-1}$, $\mathrm{Nm}(\beta) = \frac{\mathrm{Nm}(\mathfrak{c})}{\mathrm{Nm}(\mathfrak{c}'')} \mathrm{Nm}(\mathfrak{p})^{-1}$ and therefore the ast equation of diagram 3.14 becomes

$$\begin{aligned}
\varpi_\mathfrak{p}^{-\ell_\mathfrak{p}} \beta^{k+\ell}\, a_{\beta^{-1}\xi}\big((S_\mathfrak{p}^\vee f)_{\mathfrak{c}''}\big) &= \varpi_\mathfrak{p}^{-\ell_\mathfrak{p}} \beta^{-\ell} \beta^\mathsf{w}\, a_{\beta^{-1}\xi}\big((S_\mathfrak{p}^\vee f)_{\mathfrak{c}''}\big) \\
&= \varpi_\mathfrak{p}^{-\ell_\mathfrak{p}} \beta^{-\ell} \left(\frac{\mathrm{Nm}(\mathfrak{c})}{\mathrm{Nm}(\mathfrak{c}'')}\right)^\mathsf{w} \mathrm{Nm}(\mathfrak{p})^{-\mathsf{w}}\, a_{\beta^{-1}\xi}\big((S_\mathfrak{p}^\vee f)_{\mathfrak{c}''}\big) \\
&= \varpi_\mathfrak{p}^{-\ell_\mathfrak{p}} \beta^{-\ell} \left(\frac{\mathrm{Nm}(\mathfrak{c})}{\mathrm{Nm}(\mathfrak{c}'')}\right)^\mathsf{w} a_{\beta^{-1}\xi}\big((\mathrm{Nm}(\mathfrak{p})^{-\mathsf{w}} S_\mathfrak{p}^\vee f)_{\mathfrak{c}''}\big) \ .
\end{aligned}$$

Recall that $\langle\mathfrak{p}\rangle_\mathsf{w} = \big(\mathrm{Nm}(\mathfrak{p})^{-\mathsf{w}} S_\mathfrak{p}^\vee\big)^{-1}$ (see Definition 3.1.1) and consider the modular form $g := \langle\mathfrak{p}\rangle_\mathsf{w}^{-1} f$. Then the above term becomes

$$\varpi_\mathfrak{p}^{-\ell_\mathfrak{p}} \beta^{k+\ell}\, a_{\beta^{-1}\xi}\big((S_\mathfrak{p}^\vee f)_{\mathfrak{c}''}\big) = \varpi_\mathfrak{p}^{-\ell_\mathfrak{p}} \beta^{-\ell} \left(\frac{\mathrm{Nm}(\mathfrak{c})}{\mathrm{Nm}(\mathfrak{c}'')}\right)^\mathsf{w} a_{\beta^{-1}\xi}(g_{\mathfrak{c}''}) \ .$$

Now, since $T_{\mathfrak{p}}^{\circ} = T_{\mathfrak{p}}^{\vee,\circ} \circ \langle \mathfrak{p} \rangle_{\mathsf{w}}$, $T_{\mathfrak{p}}^{\circ}(g) = T_{\mathfrak{p}}^{\vee,\circ}(f)$. So Equation (3.11) becomes

$$a_{\xi}\big((T_{\mathfrak{p}}^{\circ} g)_{\mathfrak{c}}\big) = \mathrm{Nm}(\mathfrak{p})^{-1} \varpi_{\mathfrak{p}}^{-\ell_{\mathfrak{p}}} \alpha^{k+\ell}\, a_{\alpha^{-1}\xi}\big((\langle \mathfrak{p} \rangle_{\mathsf{w}}\, g)_{\mathfrak{c}'}\big) \;+\; \varpi_{\mathfrak{p}}^{-\ell_{\mathfrak{p}}} \beta^{-\ell} \left(\frac{\mathrm{Nm}(\mathfrak{c})}{\mathrm{Nm}(\mathfrak{c}'')}\right)^{\mathsf{w}} a_{\beta^{-1}\xi}(g_{\mathfrak{c}''}) \,.$$

Now it suffices to recall that, since $\alpha \mathfrak{c}' = \mathfrak{c}\mathfrak{p}$ and $\alpha \in F_+$, $\mathrm{Nm}(\alpha) = \frac{\mathrm{Nm}(\mathfrak{c})}{\mathrm{Nm}(\mathfrak{c}')} \mathrm{Nm}(\mathfrak{p})$ and use this relationship in the above equation to conclude the proof. □

Remark 3.3.10. We now study with particular attention Equation (3.18) over $\mathbb{F}$.

(a) If $k_{\tau'} > 1$ for a $\tau' \in \Sigma_{\mathfrak{p}}$ and $k_{\tau} \geq 1$ for all other $\tau \in \Sigma_{\mathfrak{p}}$, then for any $\mathfrak{c} \in \mathfrak{C}$ and $\xi \in \mathfrak{c}_+$

$$a_{\xi}^{\circ}\big((T_{\mathfrak{p}}^{\circ} f)_{\mathfrak{c}}\big) = \varpi_{\mathfrak{p}}^{-\ell_{\mathfrak{p}}} \beta^{-\ell} a_{\beta^{-1}\xi}^{\circ}(f_{\mathfrak{c}''}) \,, \tag{3.20}$$

where $\mathfrak{c}'' \in \mathfrak{C}$ and $\beta \in F^+$ are such that $\beta \mathfrak{c}'' = \mathfrak{c}\mathfrak{p}^{-1}$. This is because the $\mathfrak{p}$-valuation of the first term of Equation (3.18) is $\sum_{\tau \in \Sigma_{\mathfrak{p}}} k_{\tau} - ef$, which is positive, by our assumption on the weights k_{τ} for $\tau \in \Sigma_{\mathfrak{p}}$.

(b) For parallel weight 1 above $\mathfrak{p}$, i.e. for $k_{\tau} = 1$ for all $\tau \in \Sigma_{\mathfrak{p}}$, one will have the two terms of Equation 4.3. In fact, for $\mathfrak{c}', \mathfrak{c}'' \in \mathfrak{C}$ and $\alpha, \beta \in F^+$ are such that $\alpha \mathfrak{c}' = \mathfrak{c}\mathfrak{p}$ and $\beta \mathfrak{c}'' = \mathfrak{c}\mathfrak{p}^{-1}$. , one has that

$$\begin{aligned} v_{\mathfrak{p}}\left(\mathrm{Nm}(\mathfrak{p})^{\mathsf{w}-1} \varpi_{\mathfrak{p}}^{-\ell_{\mathfrak{p}}} \alpha^{-\ell} a_{\alpha^{-1}\xi}^{\circ}\big((\langle \mathfrak{p} \rangle_{\mathsf{w}} f)_{\mathfrak{c}'}\big)\right) &= \sum_{\tau \in \Sigma_{\mathfrak{p}}} (k_{\tau} + 2\ell_{\tau}) - ef - \sum_{\tau \in \Sigma_{\mathfrak{p}}} 2\ell_{\tau} \\ &= \sum_{\tau \in \Sigma_{\mathfrak{p}}} 1 \;- ef = 0 \,, \end{aligned}$$

and we already know that the second term has $v_{\mathfrak{p}}$-valuation equal to 0. Therefore, for any $\mathfrak{c} \in \mathfrak{C}$ and $\xi \in \mathfrak{c}_+$, the formula stays the same

$$a_{\xi}^{\circ}\big((T_{\mathfrak{p}}^{\circ} f)_{\mathfrak{c}}\big) = \mathrm{Nm}(\mathfrak{p})^{\mathsf{w}-1} \varpi_{\mathfrak{p}}^{-\ell_{\mathfrak{p}}} \alpha^{-\ell} a_{\alpha^{-1}\xi}^{\circ}\big((\langle \mathfrak{p} \rangle_{\mathsf{w}} f)_{\mathfrak{c}'}\big) + \varpi_{\mathfrak{p}}^{-\ell_{\mathfrak{p}}} \beta^{-\ell} a_{\beta^{-1}\xi}^{\circ}(f_{\mathfrak{c}''}) \,. \tag{3.21}$$

Remark 3.3.11. We take a moment to compare our formula for the action of the normalized $T_{\mathfrak{p}}^{\circ}$ operator on geometric q-expansion with known cases.

(a) $F = \mathbb{Q}$.
For $F = \mathbb{Q}$, one has that $\varpi = p$, $\alpha = p$ and $\beta = p^{-1}$. In this case, for $\mathsf{w} = k$ and for any positive integer n, one gets the very well known formula.

$$a_n(T_p f) = p^{k-1} a_{p^{-1}n}\big(\langle p \rangle f\big) + a_{pn}(f) \,.$$

(b) p inert in F.
If p is inert in F, then $\alpha = p$, $\beta = p^{-1}$ and $\varpi_p = p$. Moreover $\mathrm{Nm}(p) = \prod_{\tau \in \Sigma} p$ and $\Sigma = \Sigma_{\mathfrak{p}}$. For any $\xi \in \mathfrak{c}_+$, one has then

$$\begin{aligned} a_{\xi}\big((T_{\mathfrak{p}}^{\circ} f)_{\mathfrak{c}}\big) &= \left(\frac{\mathrm{Nm}(\mathfrak{c})}{\mathrm{Nm}(\mathfrak{c})}\right)^{\mathsf{w}} \prod_{\tau \in \Sigma} p^{k_{\tau}+2\ell_{\tau}-1} \prod_{\tau \in \Sigma} p^{-\ell_{\tau}} \prod_{\tau \in \Sigma} p^{-\ell_{\tau}} a_{p^{-1}\xi}\big((\langle p \rangle_{\mathsf{w}} f)_{\mathfrak{c}'}\big) \\ &\quad + \left(\frac{\mathrm{Nm}(\mathfrak{c})}{\mathrm{Nm}(\mathfrak{c})}\right)^{\mathsf{w}} \prod_{\tau \in \Sigma} p^{-\ell_{\tau}} \prod_{\tau \in \Sigma} p^{\ell_{\tau}} a_{p\xi}(f_{\mathfrak{c}''}) \\ &= \left(\prod_{\tau \in \Sigma} p^{k_{\tau}-1}\right) a_{p^{-1}\xi}\big((\langle p \rangle_{\mathsf{w}} f)_{\mathfrak{c}'}\big) + a_{p\xi}(f_{\mathfrak{c}}) \,. \end{aligned}$$

This matches the formula in Remark 3.14 of [ERX17a].

(c) k parallel, i.e. $k_\tau = k$ and $\ell_\tau = 0$ for all $\tau \in \Sigma$.
Let $\xi \in \mathfrak{c}_+$, then Equation (3.17) becomes

$$a_\xi\big((T_\mathfrak{p} f)_\mathfrak{c}\big) = \mathrm{Nm}(\mathfrak{p})^{k-1} \left(\frac{\mathrm{Nm}(\mathfrak{c})}{\mathrm{Nm}(\mathfrak{c}')}\right)^k a_{\alpha^{-1}\xi}\big((\langle\mathfrak{p}\rangle_\mathsf{w} f)_{\mathfrak{c}'}\big) + \left(\frac{\mathrm{Nm}(\mathfrak{c})}{\mathrm{Nm}(\mathfrak{c}'')}\right)^k a_{\beta^{-1}\xi}(f_{\mathfrak{c}''}) \,,$$

and in particular

$$a^\circ_\xi\big((T_\mathfrak{p} f)_\mathfrak{c}\big) = \mathrm{Nm}(\mathfrak{p})^{k-1} a^\circ_{\alpha^{-1}\xi}\big((\langle\mathfrak{p}\rangle_\mathsf{w} f)_{\mathfrak{c}'}\big) + a^\circ_{\beta^{-1}\xi}(f_{\mathfrak{c}''}) \,,$$

which is the known formula. We will see in the following section that this formula translates to the usual one on adelic q-expansions.

3.3.2 Adelic q-expansion

We will now end this chapter by looking at adelic q-expansions and in particular we will give the action of the $T^\circ_\mathfrak{p}$-operator in terms of adelic q-expansions. Let $\mathfrak{m}$ be an integral ideal of $\mathcal{O}_F$, then one can write $\mathfrak{m} = \xi\mathfrak{c}^{-1}$ for a unique $\mathfrak{c} \in \mathrm{Cl}^+_F$ and $\xi \in F^\times_+$. For such an ideal and a modular form f, we define

$$C(\mathfrak{m}, f) := \mathrm{Nm}(\mathfrak{c})^{-\mathsf{w}} \xi^\ell a_\xi(f_\mathfrak{c}) = \xi^\ell a^\circ_\xi(f_\mathfrak{c}), \tag{3.22}$$

where $a_\xi(f_\mathfrak{c})$ is the ξ coefficient of the q-expansion of f at $\mathfrak{c}$, and a°_ξ is the normalized geometric coefficient as defined in Definition 3.3.7.

Remark 3.3.12. These adelic coefficients obviously make sense in a field of characteristic 0 for any paritious weight, but in characteristic p, these coefficients make sense only in parallel weight, i.e. when $k_\tau = k$ and $\ell_\tau = 0$ for all $\tau \in \Sigma$. This is the reason why we are obliged to work with the geometric coefficients when dealing with partial weight. For the parallel case, the adelic coefficients are more convenient because the formulas are more compact and clean.

Proposition 3.3.13. *The above definition is independent of the choice of* ξ *and of the choice of representative* $\mathfrak{c}$.

Proof. Another choice of ξ is $\varepsilon\xi$ for $\varepsilon \in \mathcal{O}^\times_{F,+}$. For such an element we have that $a_{\varepsilon\xi} = \varepsilon^{-\ell} a_\xi$. Therefore

$$\mathrm{Nm}(\mathfrak{c})^{-\mathsf{w}}\ (\varepsilon\xi)^\ell\ a_{\varepsilon\xi}(f_\mathfrak{c}) = \mathrm{Nm}(\mathfrak{c})^{-\mathsf{w}}\ (\varepsilon\xi)^\ell\ \varepsilon^{-\ell} a_\xi(f_\mathfrak{c}) = \mathrm{Nm}(\mathfrak{c})^{-\mathsf{w}}\ \xi^\ell\ a_\xi(f_\mathfrak{c}) \,.$$

Another choice of a class representative for $\mathfrak{c}$ is $\nu\mathfrak{c}$ for $\nu \in F^\times_+$. By Proposition **??**, one has that

$$\mathrm{Nm}(\nu\mathfrak{c})^{-\mathsf{w}} (\nu\xi)^\ell a_\xi(f_{\nu\mathfrak{c}}) = \mathrm{Nm}(\nu\mathfrak{c})^{-\mathsf{w}} (\nu\xi)^\ell \nu^{k+\ell} a_\xi(f_\mathfrak{c}) = \mathrm{Nm}(\mathfrak{c})^{-\mathsf{w}}\ \xi^\ell\ a_\xi(f_\mathfrak{c}) \,.$$

□

Remark 3.3.14. Our definition of $C(\mathfrak{m}, f)$ differs from the one of Shimura ([Shi78, Equation 2.24]) in the normalization factor. In fact, Shimura normalizes the adelic coefficients by $\mathrm{Nm}(\mathfrak{c})^{-k_0/2}$, where k_0 is the maximum of the $(k_\tau)_{\tau\in\Sigma}$. This difference is due to the fact that Shimura considers forms that are on the sheaf $\omega^k \otimes \delta^{k/2}$, while here we consider independent powers ℓ on the determinant sheaf.

Corollary 3.3.15. *Let the weight (k,ℓ) be paritious, i.e. $k_\tau + 2\ell_\tau = \mathsf{w}$ for all $\tau \in \Sigma$ and let $f \in \mathrm{H}^0(\mathrm{Sh}, \omega^{k,\ell})$ be a HMF. Then in $K = \mathrm{Frac}(\mathcal{O})$ one has that*

$$C(\mathfrak{m}, T_{\mathfrak{q}}\ f) = \mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1} C\big(\mathfrak{m}\mathfrak{q}^{-1}, \langle \mathfrak{q} \rangle_{\mathsf{w}}\ f\big) + C(\mathfrak{m}\mathfrak{q}, f)\ ,$$

and

$$C(\mathfrak{m}, T^\circ_{\mathfrak{p}}\ f) = \varpi_p^{-\ell_{\mathfrak{p}}} \Big(\mathrm{Nm}(\mathfrak{p})^{\mathsf{w}-1} C\big(\mathfrak{m}\mathfrak{p}^{-1}, \langle \mathfrak{p} \rangle_{\mathsf{w}}\ f\big) + C(\mathfrak{m}\mathfrak{p}, f)\Big)\ .$$

In particular, for parallel weight $(k,0)$, one has that in $\mathcal{O}$

$$C(\mathfrak{m}, T_{\mathfrak{q}}\ f) = \mathrm{Nm}(\mathfrak{q})^{k-1} C\big(\mathfrak{m}\mathfrak{q}^{-1}, \langle \mathfrak{q} \rangle_{k}\ f\big) + C(\mathfrak{m}\mathfrak{q}, f)\ ,$$

for any prime ideal $\mathfrak{q} \subset$ coprime with $p\mathfrak{n}$ and for $\mathfrak{q} = \mathfrak{p}$.

Proof. Consider Equation (3.19):

$$a^\circ_\xi\big((T_{\mathfrak{q}} f)_{\mathfrak{c}}\big) = \mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1} \alpha^{-\ell} a^\circ_{\alpha^{-1}\xi}\Big(\big(\langle \mathfrak{q} \rangle_{\mathsf{w}} f\big)_{\mathfrak{c}'}\Big) + \beta^{-\ell} a^\circ_{\beta^{-1}\xi}(f_{\mathfrak{c}''})\ ,$$

where $\alpha, \beta \in F_+$ are such that $\alpha\mathfrak{c}' = \mathfrak{c}\mathfrak{q}$ and $\beta\mathfrak{c}'' = \mathfrak{c}\mathfrak{q}^{-1}$. Let us remark that for an integral ideal $\mathfrak{m} \subset \mathcal{O}_F$, such that $\mathfrak{m} = \xi\mathfrak{c}^{-1}$, then

$$\mathfrak{m}\mathfrak{q}^{-1} = \xi\alpha^{-1}(\mathfrak{c}')^{-1} \quad \text{and} \quad \mathfrak{m}\mathfrak{q} = \xi\beta^{-1}(\mathfrak{c}'')^{-1}\ ,$$

and therefore by definition

$$\begin{aligned} C(\mathfrak{m}\mathfrak{q}^{-1}, \cdot) &= \mathrm{Nm}(\mathfrak{c}')^{-\mathsf{w}} (\alpha^{-1}\xi)^\ell a_{\alpha^{-1}\xi}(\cdot) \\ C(\mathfrak{m}\mathfrak{q}, \cdot)\ &= \mathrm{Nm}(\mathfrak{c}'')^{-\mathsf{w}} (\beta^{-1}\xi)^\ell a_{\beta^{-1}\xi}(\cdot) \end{aligned}$$

Putting everything together, one gets that

$$\begin{aligned} C(\mathfrak{m}, T_{\mathfrak{q}}\ f) &= \xi^\ell a^\circ_\xi\big((T_{\mathfrak{q}} f)_{\mathfrak{c}}\big) \\ &= \mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1} \xi^\ell \alpha^{-\ell} a^\circ_{\alpha^{-1}\xi}\big(\langle \mathfrak{q} \rangle_{\mathsf{w}}\ f\big)_{\mathfrak{c}'} + \xi^\ell \beta^{-\ell} a_{\beta^{-1}\xi}(f_{\mathfrak{c}''}) \\ &= \mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1} C\big(\mathfrak{m}\mathfrak{q}^{-1}, \langle \mathfrak{q} \rangle_{\mathsf{w}}\ f\big) + C(\mathfrak{m}\mathfrak{q}, f)\ . \end{aligned}$$

The same arguments work for the normalized Hecke operator $T^\circ_{\mathfrak{p}} = \varpi_{\mathfrak{p}}^{-\ell_{\mathfrak{p}}} T_{\mathfrak{p}}$, using equation (3.18).
For parallel weight $(k,0)$, the adelic coefficients $C(\mathfrak{m}, T_{\mathfrak{q}} f)$ are by definition (see Equation (3.22)) given by $a^\circ_\xi((T_{\mathfrak{q}} f)_{\mathfrak{c}})$, where $\xi \in F_+^\times$ is an element such that $\mathfrak{m}\mathfrak{c} = (\xi)$. So in particular, these coefficients are integral, since $a^\circ_\xi((T_{\mathfrak{q}} f)_{\mathfrak{c}}) \in \mathcal{O}$. The formula follows from the adelic formula for $T_{\mathfrak{q}}$. Let us also point out that for parallel weight $T^\circ_{\mathfrak{p}} = T_{\mathfrak{p}}$. ☐

Remark 3.3.16. The formulas obtained in Corollary 3.3.15 are a generalization of previous known formulas for parallel weight and non-normalized $T_{\mathfrak{p}}$ Hecke operator, found for example in [DW18, Theorem 1.2].

Chapter 4

Unramifiedness of Galois representations modulo ϖ

In this chapter we will only work with paritious weight forms, i.e. Hilbert modular forms of weights $(k,\ell) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}^{\Sigma}$. such that $k_\tau + 2\ell_\tau$ is independent of τ, i.e. there exists a $\mathsf{w} \in \mathbb{Z}$ such that $k_\tau + 2\ell_\tau = \mathsf{w}$ for all $\tau \in \Sigma$. It is clear that it is enough to consider a couple $(k,\mathsf{w}) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}$ to describe such weights. Therefore in what follows we will denote the sheaf of differentials of paritious weight (k,w) by

$$\omega^{(k,\mathsf{w})} := \bigotimes_{\tau\in\Sigma} \left(\omega_\tau^{\otimes k_\tau} \otimes_{\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}} \delta_\tau^{\otimes(\mathsf{w}-k_\tau)/2} \right) .$$

Definition 4.0.1. We denote by $\mathcal{M}_{k,\mathsf{w}}(\mathfrak{n};R) := \mathrm{H}^0\big(\mathrm{Sh}_R, \omega_R^{(k,\mathsf{w})}\big)$ the R-module of *Hilbert modular forms of level* $\mathfrak{n}$ *and paritious weight* (k,w) *with coefficients over an* $\mathcal{O}$*-algebra* R, and by $\mathcal{S}_{k,\mathsf{w}}\big(\mathfrak{n};R\big) := \mathrm{H}^0\big(\mathrm{Sh}_R, \omega_R^{(k,\mathsf{w})}(-\mathrm{D})\big)$ the submodule of *cuspidal forms.* (see Chapter 2 for more details.)

Recall that we have Hecke operators $T_\mathfrak{q}$ for $\mathfrak{q} \subset \mathcal{O}_F$ a prime not dividing $p\mathfrak{n}$ acting on $\mathcal{M}_{k,\mathsf{w}}(\mathfrak{n};\mathcal{O})$. Moreover, in Chapter 3, we constructed normalized diamond operators $\langle\mathfrak{q}\rangle_\mathsf{w}$ for any prime ideal $\mathfrak{q} \subset \mathcal{O}_F$, and we have recalled the construction by Emerton-Reduzzi-Xiao of a normalized Hecke operator $T_\mathfrak{p}^\circ$ for $\mathfrak{p} \subset \mathcal{O}_F$ a prime above p.
Our goal is to show the following generalization to non-parallel paritious weight 1 Hilbert modular forms of results of Dimitrov-Wiese ([DW18, Theorem 1.1]) and of Emerton-Reduzzi-Xiao ([ERX17a, Theorem 1.1]).

Theorem 4.0.2. *Let* $\mathfrak{p}$ *be a fixed prime of* F *above* p*. Let* $(k,1)$ *be a paritious weight such that* $k_\tau = 1$ *for all* $\tau \in \Sigma_\mathfrak{p}$*. Let* $f \in \mathcal{S}_{k,1}(\mathfrak{n},\mathbb{F})$ *be a cuspidal Hilbert modular form and assume that* f *is common eigenvector for the Hecke operators* $T_\mathfrak{q}$ *and* $\langle\mathfrak{q}\rangle_1$ *for all* $\mathfrak{q}$ *outside a finite set* S *of primes of* F*, containing* $\{v : v \text{ a place of } F, v \neq \mathfrak{p} \text{ and } v|p\mathfrak{n}\}$*. Then there exists a continuous semi-simple representation*

$$\rho_f : G_F \longrightarrow \mathrm{GL}_2(\mathbb{F}) ,$$

which is unramified at all primes $\mathfrak{q}$ *not dividing* $p\mathfrak{n}$ *and at* $\mathfrak{q} = \mathfrak{p}$*, and is such that the trace of* $\rho_f(\mathrm{Frob}_\mathfrak{q})$ *equals the eigenvalue of* $T_\mathfrak{q}$ *on* f *for all such primes* $\mathfrak{q}$*.*

In order to prove this theorem we will need many ingredients. Firstly, we will need a way to lift modulo ϖ modular forms to characteristic 0. As explained in Chapter 2, one can only hope to lift paritious weight forms, this explains why we restrict to work with paritious weight forms and not with general partial weight forms. Lifting forms will be achieved through an exceptional sheaf of paritious weight 0, which will be described in the next section. This sheaf is a variation of the exceptional sheaf defined in [RX17, Lemma 2.7]. Moreover, we will work in the generality of Hilbert modular form modulo ϖ^m, in order to lay the fundation to extend the above result to the entire Hecke algebra (see section 4.4). Secondly, we will need to work with the generalized partial Hasse invariants as defined by Reduzzi and Xiao in [RX17, Section 3]. In Section 4.2, we will recall and prove some of their properties. Finally, we will use the doubling method of Wiese ([Wie14]) and follow the strategy of Dimitrov-Wiese ([DW18]) to finish the proof to prove our result. This will be done in the remaining sections.

4.1 Lifting modulo ϖ^m Hilbert Modular Forms

It is known that Hilbert modular forms mod ϖ^m of low weight are not necessarily all liftable in characteristic 0. However, if one can embed those, e.g. via multiplication by powers of partial Hasse invariants, as a Hecke stable subspace in liftable weight, then by a result of Deligne and Serre ([DS74, Lemme 6.11]) the corresponding systems of eigenvalues would lift as well and thus one can attach Galois representations to the original mod ϖ^m eigenforms. In particular, one knows that for parallel weights, there always exists a big enough weight where the forms can be lifted in characteristic 0, see [DDW19, Lemma 2.2]. This is not the case for partial weight forms. Because of the description of the geometric q expansion (2.4.1), one can only hope to be able to lift cuspforms for some "big" enough weights. This is the object of Proposition 4.6 of [RX17], where Reduzzi and Xiao prove that all weights in a "sufficiently positive direction" are liftable weights for cuspforms. This direction is given by a specific weight, denoted ex, which looks mysterious at first. However this direction can be given a heuristic explanation by works of Diamond and Kassaei ([DK17], [DK20]) since moving in the direction called ex by Reduzzi-Xiao brings form in what Diamond and Kassaei call the *minimal cone*, where forms are liftable. Let us recall the definition of Diamond and Kassaei (see [DK17] and [DK20] for unramified p) of the *minimal cone.*

Definition (Definition 3.3.1)**.** We say that a weight vector $k \in \mathbb{Z}^\Sigma$ belongs to the *minimal cone*, denoted C^{min}, if for every $\mathfrak{p}|p$:

- $k_{\tau_{\mathfrak{p},j}^{(i+1)}} \geq k_{\tau_{\mathfrak{p},j}^{(i)}}$ for all $j = 1, \ldots, f_\mathfrak{p}$ and $i = 1, \ldots, e_\mathfrak{p} - 1$;
- $pk_{\tau_{\mathfrak{p},j}^{(1)}} \geq k_{\tau_{\mathfrak{p},j-1}^{(e_\mathfrak{p})}}$ for all $j = 1, \ldots f_\mathfrak{p}$.

In what follows, we will construct an exceptional sheaf, along the lines of Reduzzi-Xiao ([RX17, Lemma 2.7]), show some of its properties and finally use this sheaf to prove a lifting lemma for partious weight HMFs.

4.1.1 Exceptional Sheaf

Inspired by the exceptional sheaf of differentials defined by Reduzzi and Xiao in ([RX17, Lemma 2.7]), we set the *exceptional weight* to be the weight vector $\mathrm{ex} \in \mathbb{Z}^\Sigma$ such that $\mathrm{ex}_{\tau_{\mathfrak{p},j}^{(i)}} = 2(2i - e_\mathfrak{p} - 1)$ for all $\mathfrak{p}|p$, $j = 1, \ldots, f_\mathfrak{p}$ and $i = 1, \ldots, e_\mathfrak{p}$. In particular, we will call *exceptional sheaf* on $\mathrm{Y}_{\mathbb{F}}^{\mathrm{PR}}$

the line bundle:

$$\dot{\omega}_{\mathbb{F}}^{(\mathrm{ex},0)} = \bigotimes_{\mathfrak{p}|p} \bigotimes_{j=1}^{f_\mathfrak{p}} \bigotimes_{i=1}^{e_\mathfrak{p}} \left(\dot{\omega}_{\tau_{\mathfrak{p},j}^{(i)},\mathbb{F}}^{\otimes 2(2i-e_\mathfrak{p}-1)} \otimes_{\mathcal{O}_{\mathrm{Y}_{\mathbb{F}}^{\mathrm{PR}}}} \dot{\delta}_{\tau_{\mathfrak{p},j}^{(i)},\mathbb{F}}^{\otimes(e_\mathfrak{p}+1-2i)} \right), \tag{4.1}$$

which descends to a line bundle on $\mathrm{Sh}_{\mathbb{F}}^{\mathrm{PR}}$. We now proceed to adapt results and proofs of Reduzzi-Xiao [RX17] to our exceptional sheaf.

Lemma 4.1.1. *The line bundle $\dot{\omega}_{\mathbb{F}}^{(\mathrm{ex},0)}$ is relatively ample with respect to the natural projection $\mathrm{Y}_{\mathbb{F}}^{\mathrm{PR}} \to \mathrm{Y}_{\mathbb{F}}^{\mathrm{DP}}$.*

Proof. This follows exactly from the same argument as Lemma 2.7 of [RX17]. □

Lemma 4.1.2. *The line bundle $\dot{\omega}_{\mathbb{F}}^{(\mathrm{ex},0)}$ defined over $\mathrm{Y}_{\mathbb{F}}^{\mathrm{PR,tor}}$ descends to a line bundle, denoted $\dot{\omega}_{\mathbb{F},\mathrm{min}}^{(\mathrm{ex},0)}$, over the minimal compactification $\mathrm{Y}_{\mathbb{F}}^{\mathrm{PR,min}}$, which is relatively ample with respect to the natural projection $\mathrm{Y}_{\mathbb{F}}^{\mathrm{PR,min}} \to \mathrm{Y}_{\mathbb{F}}^{\mathrm{DP,min}}$.*

Proof. Since the sheaves $\dot{\delta}_\tau$ are all trivial over $\mathcal{O}_{\mathrm{Y}_{\mathbb{F}}^{\mathrm{PR}}}$(see Equation 2.6), we will be interested only in the k-part of the sheaves, i.e. in sheaves of the form $\dot{\omega}^k_{\mathrm{Y}^{\mathrm{PR,tor}}} = \bigotimes_{\tau\in\Sigma} \dot{\omega}_\tau^{k_\tau}$. By works of [Rap78] and [Cha90], we know that for $k \in \mathbb{Z}^\Sigma$, the sheaf $\dot{\omega}^k_{\mathrm{Y}^{\mathrm{PR,tor}}}$ descends to the minimal compatification $\mathrm{Y}^{\mathrm{PR,min}}$ if and only if k is parallel. However, the situation is different on the special fiber, $\pi_{\mathbb{F}} : \mathrm{Y}_{\mathbb{F}}^{\mathrm{PR,tor}} \to \mathrm{Y}_{\mathbb{F}}^{\mathrm{PR,min}}$. Going through the proof of [Dim04, Théorème 8.6 part (vi)], for an $\mathcal{O}$-algebra R, one sees that in general the sheaf $\dot{\omega}^k_{\mathrm{Y}_R^{\mathrm{PR,tor}}}$ descends to an invertible sheaf on $\mathrm{Y}_R^{\mathrm{PR,min}}$ if and only if it can be trivialized on $S^\wedge_{\Sigma\mathcal{C}}/\mathcal{O}^\times_{F,\mathfrak{n}} \times \mathrm{Spec}(R)$, in the sense that it is free of rank 1 on the structure sheaf $\mathcal{O}_{\mathrm{Y}_R^{\mathrm{PR,tor}}}$. Consider a cusp $\mathcal{C} = (\mathfrak{a}, \mathfrak{b}, H, i, j, \lambda, \alpha)$ and let $S^\wedge_{\Sigma\mathcal{C}}$ be as in [Dim04, Théorème 7.2], then the pullback of $\dot{\omega}_{\mathrm{Y}_R^{\mathrm{PR,tor}}}$ to $S^\wedge_{\Sigma\mathcal{C}} \times \mathrm{Spec}(R)$ is canonically trivial and isomorphic to

$$(\mathfrak{a} \otimes \mathcal{O})^k \otimes_{\mathcal{O}} R \otimes_{\mathcal{O}} \mathcal{O}_{S^\wedge_{\Sigma\mathcal{C}}}, \tag{4.2}$$

which in particular tells us how a unit $u \in \mathcal{O}^\times_{F,\mathfrak{n}}$ acts on this sheaf, i.e. via multiplication by $u^{k/2} := \prod_{\tau\in\Sigma} \tau(u)^{k_\tau/2}$. In fact,

$$\mathrm{H}^0\left(S^\wedge_{\Sigma\mathcal{C}}/\mathcal{O}^\times_{F,\mathfrak{n}} \times \mathrm{Spec}(R), \dot{\omega}^k_{\mathrm{Y}_R^{\mathrm{PR,tor}}}\right) = \left\{ \sum_{\xi\in X^+\cup\{0\}} a_\xi q^\xi : a_\xi \in R,\ a_{u^2\xi} = u^k a_\xi \text{ for all } u \in \mathcal{O}^\times_{F,\mathfrak{n}} \right\},$$

which is a projective module, but not free of rank 1. Actually, we want this module to be isomorphic to

$$\mathrm{H}^0\left(S^\wedge_{\Sigma\mathcal{C}}/\mathcal{O}^\times_{F,\mathfrak{n}} \times \mathrm{Spec}(R), \mathcal{O}_{\mathrm{Y}_R^{\mathrm{PR,tor}}}\right) = \left\{ \sum_{\xi\in X^+\cup\{0\}} a_\xi q^\xi : a_\xi \in R,\ a_{u\xi} = a_\xi \text{ for all } u \in \mathcal{O}^\times_{F,\mathfrak{n}} \right\},$$

i.e. the line bundle $\omega^k_{\mathrm{Y}_R^{\mathrm{PR,tor}}}$ will descend to $\mathrm{Y}_R^{\mathrm{PR,min}}$ if and only if $u^{k/2}$ act trivially in R. Therefore, in order to see if the line bundle $\dot{\omega}^{\mathrm{ex}}$ descends to the minimal compactification $\mathrm{Y}_{\mathbb{F}}^{\mathrm{PR,min}}$, it suffices to verify that $u^{\mathrm{ex}/2} = 1$ in $\mathbb{F}$. In fact,

$$\begin{aligned} u^{\mathrm{ex}/2} &= \prod_{\tau\in\Sigma} \tau(u)^{\mathrm{ex}_\tau/2} \\ &= \prod_{\mathfrak{p}|p} \prod_{j=1}^{f_\mathfrak{p}} \prod_{i=1}^{e_\mathfrak{p}} \left(\tau_{\mathfrak{p},j}^{(i)}(u)\right)^{2i-e_\mathfrak{p}-1}, \end{aligned}$$

and, since $\tau_{\mathfrak{p},j}^{(i)} \equiv \tau_{\mathfrak{p},j}^{(i+1)} \mod \varpi$ for all $i \in \{1, \dots e_\mathfrak{p} - 1\}$, the above product in $\mathbb{F}$ becomes:

$$u^{\mathrm{ex}/2} = \prod_{\mathfrak{p}|p} \prod_{j=1}^{f_\mathfrak{p}} \left(\tau_{\mathfrak{p},j}^{(1)}(u)\right)^{\sum_{i=1}^{e_\mathfrak{p}}(2i-e_\mathfrak{p}-1)} = 1,$$

since $\sum_{i=1}^{e_\mathfrak{p}}(2i - e_\mathfrak{p} - 1) = 0$. The relative ampleness of $\dot{\omega}_{\mathbb{F},\min}^{(\mathrm{ex},0)}$ with respect to the natural map $\mathrm{Y}_\mathbb{F}^{\mathrm{PR},\min} \to \mathrm{Y}_\mathbb{F}^{\mathrm{DP},\min}$ follows by the previous lemma. □

4.1.2 Lifting Lemma

In order to achieve a lifting lemma, we want to transform the exceptional line bundle into an ample line bundle on the minimal compactification $\mathrm{Y}_\mathbb{F}^{\mathrm{PR},\min}$. We will denote by $\mathtt{t}$ the element $(1, \dots, 1)$ in $\mathbb{Z}^\Sigma$.

Lemma 4.1.3. *There exists a positive integer* $\mathtt{N}_0 \in \mathbb{Z}_{>0}$ *such that for any* $\mathtt{N} \geq \mathtt{N}_0$*, the weight vector* $\mathtt{Nt} + \mathrm{ex}$ *lies in the minimal cone* $C^{\min}$.

Proof. This follows immediately from the definition of the minimal cone, Definition 3.3.1, and from the definition of the weight vector ex. □

We fix once and for all such an integer $\mathtt{N}_0$.

Lemma 4.1.4. *There exists an even integer* $\mathtt{N} \geq \mathtt{N}_0$ *such that the line bundle* $\dot{\omega}_\mathbb{F}^{(\mathtt{Nt}+\mathrm{ex},0)}$ *on* $\mathrm{Y}_\mathbb{F}^{\mathrm{PR},\mathrm{tor}}$ *descends to an ample line bundle on the minimal compactification* $\mathrm{Y}_\mathbb{F}^{\mathrm{PR},\min}$*. Similarly, for the same* $\mathtt{N}$*, the line bundle* $\omega_\mathbb{F}^{(\mathtt{Nt}+\mathrm{ex},0)}$ *descends to an ample line bundle on the minimal compactification* $\mathrm{Sh}_\mathbb{F}^{\mathrm{PR},\min}$.

Proof. This follows from the exact same argument as in [RX17, Lemma 4.5], using relative ampleness from Lemma 4.1.2.

□

We fix once and for all an even integer $\mathtt{N}$ **as in Lemma 4.1.4**, i.e. such that the line bundle $\omega_\mathbb{F}^{(\mathtt{Nt}+\mathrm{ex},\,0)}$ descends to an ample line bundle on the minimal compactification $\mathrm{Sh}_\mathbb{F}^{\mathrm{PR},\min}$.

Lemma 4.1.5. *For any paritious weight* $(k, \mathsf{w}) \in \mathbb{Z}^\Sigma \times \mathbb{Z}$*, there is an integer* $r_0 = r_0(k, \mathsf{w})$ *such that for any* $r \geq r_0$ *and any* $i > 0$ *one has*

$$\mathrm{H}^i\left(\mathrm{Sh}^{\mathrm{PR},\mathrm{tor}}, \omega^{(k+r(\mathtt{Nt}+\mathrm{ex}),\mathsf{w})}(-\mathrm{D})\right) = 0\,.$$

Proof. This follows form the exact same argument as in [ERX17b, Lemma 4.2.2], using the ampleness from Lemma 4.1.4. □

We now have all the ingredients to prove the following Lifting Lemma.

Lemma 4.1.6 (Lifting Lemma). *For any paritiotus weight* $(k, \mathsf{w}) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}$*, there exists an integer* $r_0 = r_0(k, \mathsf{w})$ *such that for any* $r \geq r_0$ *there is a natural Hecke equivariant isomorphism:*

$$\mathcal{S}_{k+r\cdot(\mathtt{Nt}+\mathrm{ex}),\mathsf{w}}(\mathfrak{n}; \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m\mathcal{O} \xrightarrow{\sim} \mathcal{S}_{k+r\cdot(\mathtt{Nt}+\mathrm{ex}),\mathsf{w}}(\mathfrak{n}; \mathcal{O}/\varpi^m\mathcal{O}).$$

Proof. Let us set $\tilde{k} := k + r \cdot (\mathtt{Nt} + \mathrm{ex})$ and $R_m := \mathcal{O}/\varpi^m\mathcal{O}$. Recall that we denote by D the divisor of the cusps. The sheaf $\omega^{(\tilde{k},\mathsf{w})}$ is a locally free $\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}$-module of rank 1, and therefore one has a short exact sequence of sheaves on $\mathrm{Sh}^{\mathrm{tor}}$

$$0 \longrightarrow \omega^{(\tilde{k},\mathsf{w})}(-\mathrm{D}) \xrightarrow{\cdot\varpi_{\mathfrak{p}}^m} \omega^{(\tilde{k},\mathsf{w})}(-\mathrm{D}) \longrightarrow \omega_{R_m}^{(\tilde{k},\mathsf{w})}(-\mathrm{D}) \longrightarrow 0,$$

which induces a long exact sequence in cohomology

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(\mathrm{Sh}, \omega^{(\tilde{k},\mathsf{w})}(-\mathrm{D})) \xrightarrow{\cdot\varpi_{\mathfrak{p}}^n} \mathrm{H}^0(\mathrm{Sh}, \omega^{(\tilde{k},\mathsf{w})}(-\mathrm{D})) \longrightarrow \mathrm{H}^0(\mathrm{Sh}, \omega_{R_m}^{(\tilde{k},\mathsf{w})}(-\mathrm{D})) \longrightarrow \\ \longrightarrow \mathrm{H}^1(\mathrm{Sh}, \omega^{(\tilde{k},\mathsf{w})}(-\mathrm{D})) \end{aligned}$$

Now for $\tilde{k}$ as defined as above, $\mathrm{H}^1(\mathrm{Sh}, \omega^{(\tilde{k},\mathsf{w})}(-\mathrm{D})) = 0$ by Lemma 4.1.5, and by definition of cusp forms (see Definition 4.0.1), one has a short exact sequence of $\mathcal{O}$-modules

$$0 \longrightarrow \mathcal{S}_{\tilde{k},\mathsf{w}}(\mathfrak{n}; \mathcal{O}) \xrightarrow{\cdot\varpi_{\mathfrak{p}}^n} \mathcal{S}_{\tilde{k},\mathsf{w}}(\mathfrak{n}; \mathcal{O}) \longrightarrow \mathcal{S}_{\tilde{k},\mathsf{w}}(\mathfrak{n}; \mathcal{O}/\varpi^m\mathcal{O}) \longrightarrow 0\ ,$$

which yields the desired result.

□

4.2 Generalized partial Hasse invariants

In this section, we will recall the generalized partial Hasse invariants defined by Reduzzi and Xiao (see [RX17, Section 3.1]) and we will use them to construct a form, whose weight is in the "liftable direction".

Definition 4.2.1 (Section 3.1 [RX17]). For every $\tau \in \Sigma$, there exists a Hilbert modular form

$$h_\tau \in \begin{cases} \mathrm{H}^0\left(\mathrm{Sh}_{\mathbb{F}}, \omega^{\otimes -1}_{\tau^{(1)}_{\mathfrak{p},j},\mathbb{F}} \otimes \omega^{\otimes p}_{\tau^{(e_\mathfrak{p})}_{\mathfrak{p},j},\mathbb{F}}\right), & \text{if } \tau = \tau^{(1)}_{\mathfrak{p},j} \\ \\ \mathrm{H}^0\left(\mathrm{Sh}_{\mathbb{F}}, \omega^{\otimes -1}_{\tau^{(i)}_{\mathfrak{p},j},\mathbb{F}} \otimes \omega^{\otimes 1}_{\tau^{(i-1)}_{\mathfrak{p},j},\mathbb{F}}\right), & \text{if } \tau = \tau^{(i)}_{\mathfrak{p},j} \text{ for } i \neq 1 \end{cases}$$

called the *generalized partial Hasse invariant*. We will denote by w^τ the weight of the generalized partial Hasse invariant h_τ.

In particular, the generalized partial Hasse is not a paritious weight form, and therefore it cannot be lifted to characteristic 0. However, using the trivializations ([RX17, 3.2.1])

$$\delta_{\tau^{(1)}_{\mathfrak{p},j},\mathbb{F}} \otimes \delta^{\otimes(-p)}_{\tau^{(e_\mathfrak{p})}_{\mathfrak{p},j-1},\mathbb{F}} \simeq \mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}} \quad \text{and} \quad \delta_{\tau^{(i)}_{\mathfrak{p},j},\mathbb{F}} \otimes \delta^{\otimes -1}_{\tau^{(i-1)}_{\mathfrak{p},j},\mathbb{F}} \simeq \mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}\ ,$$

one can view

$$h_\tau^2 \in \begin{cases} \mathrm{H}^0\left(\mathrm{Sh}_{\mathbb{F}}, \omega^{\otimes -2}_{\tau^{(1)}_{\mathfrak{p},j},\mathbb{F}} \otimes \omega^{\otimes 2p}_{\tau^{(e_\mathfrak{p})}_{\mathfrak{p},j},\mathbb{F}} \otimes \delta_{\tau^{(1)}_{\mathfrak{p},j},\mathbb{F}} \otimes \delta^{\otimes -p}_{\tau^{(e_\mathfrak{p})}_{\mathfrak{p},j-1},\mathbb{F}}\right), & \text{if } \tau = \tau^{(1)}_{\mathfrak{p},j} \\ \mathrm{H}^0\left(\mathrm{Sh}_{\mathbb{F}}, \omega^{\otimes -2}_{\tau^{(i)}_{\mathfrak{p},j},\mathbb{F}} \otimes \omega^{\otimes 2}_{\tau^{(i-1)}_{\mathfrak{p},j},\mathbb{F}} \otimes \delta_{\tau^{(i)}_{\mathfrak{p},j},\mathbb{F}} \otimes \delta^{\otimes -1}_{\tau^{(i-1)}_{\mathfrak{p},j},\mathbb{F}}\right), & \text{if } \tau = \tau^{(i)}_{\mathfrak{p},j} \text{ for } i \neq 1 \end{cases}$$

as a paritious weight $\mathsf{w} = 0$ Hilbert modular forms. In particular, $h_\tau^2 \in \mathrm{H}^0(\mathrm{Sh}_{\mathbb{F}}, \omega_{\mathbb{F}}^{(2w^\tau,0)})$.

By Lemma 1.4 of [DDW19], for any $\tau \in \Sigma$, the generalized partial Hasse invariant h_τ has geometric q-expansion equal to 1 at each cusp $\infty(\mathfrak{c})$. Therefore, also h_τ^2 has geometric q-expansion equal to 1 at every infinity cusp $\infty(\mathfrak{c})$. So multiplying Hilbert modular forms by these elements will not change their q-expansions.

4.2.1 Working over $\mathbb{F}$

The following lemma shows that there exists a product of generalized partial Hasse invariants modulo p that lives in the direction of the liftable weight of the previous section.

Lemma 4.2.2. *There exist a product of partial Hasse invariants h_{ex} lying in $\mathrm{H}^0\left(\mathrm{Sh}_{\mathbb{F}}, \omega_{\mathbb{F}}^{(\kappa_{\mathrm{ex}},0)}\right)$, where $\kappa_{\mathrm{ex}} = (p-1)(\mathtt{Nt} + \mathrm{ex})$. Moreover, for any $\mathfrak{c} \in \mathfrak{C}$, the q-expansion of h_{ex} at the cusp $\infty(\mathfrak{c})$ is* 1.

Proof. This statement can be proven via linear algebra methods on the weights of the generalized partial Hasse invariants of Definition 4.2.1. Let $\mathfrak{p}|p$ be a fixed prime ideal of $\mathcal{O}_F$ and let $W_\mathfrak{p}$ be the matrix of weights of generalized partial Hasse invariants h_τ for $\tau \in \Sigma_\mathfrak{p}$. Then $W_\mathfrak{p}$ is of the form

$$W_\mathfrak{p} = \left(\begin{array}{cccc|cccc|cccc}
-1 & 1 & & & & & & & & & & \\
 & \ddots & \ddots & & & & & & & & & \\
 & & \ddots & 1 & & & & & & & & \\
 & & & -1 & p & & & & & & & \\
\hline
 & & & & -1 & 1 & & & & & & \\
 & & & & & \ddots & \ddots & & & & & \\
 & & & & & & \ddots & 1 & & & & \\
 & & & & & & & -1 & p & & & \\
\hline
 & & & & & & & & -1 & 1 & & \\
 & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & \ddots & 1 \\
p & & & & & & & & & & & -1
\end{array}\right)$$

where each block has dimension $e_\mathfrak{p} \times e_\mathfrak{p}$ and the matrix has dimension $e_\mathfrak{p} f_\mathfrak{p}$. Let $\mathbf{1}_\mathfrak{p}$ denote the vector of length $e_\mathfrak{p} f_\mathfrak{p}$ whose entries are all 1. Let us write the vector of weights corresponding to the sheaf $\omega_{\mathbb{F}}^{(\mathrm{ex},0)}$ at the prime ideal $\mathfrak{p}$ as

$$\mathrm{ex}_{\mathfrak{p}} = \begin{pmatrix} 2(-e_{\mathfrak{p}}+1) \\ 2(-e_{\mathfrak{p}}+3) \\ \vdots \\ 2(e_{\mathfrak{p}}-1) \\ \hline 2(-e_{\mathfrak{p}}+1) \\ \vdots \\ \vdots \\ 2(e_{\mathfrak{p}}-1) \\ \hline 2(-e_{\mathfrak{p}}+1) \\ 2(-e_{\mathfrak{p}}+3) \\ \vdots \\ 2(e_{\mathfrak{p}}-1) \end{pmatrix} .$$

So we want to solve for all $\mathfrak{p}|p$ the linear system $2W_{\mathfrak{p}} \cdot x = (\mathrm{ex}_{\mathfrak{p}} + \mathtt{N}\mathbf{1}_{\mathfrak{p}})(p-1)$, which admits a $\mathbb{Z}$ solution. In fact, Gauss reduction gives rise to the following equation:

$$\begin{aligned}(p^{f_{\mathfrak{p}}}-1)x_{e_{\mathfrak{p}}f_{\mathfrak{p}}} = &\Bigg((e_{\mathfrak{p}}-1) + p \cdot \sum_{i=1}^{e_{\mathfrak{p}}} \big(2i - e_{\mathfrak{p}} - 1\big) + p^2 \cdot \sum_{i=1}^{e_{\mathfrak{p}}} \big(2i - e_{\mathfrak{p}} - 1\big) + \dots \\ &\dots + p^{f_{\mathfrak{p}}} \cdot \sum_{i=1}^{e_{\mathfrak{p}}-1} \big(2i - e_{\mathfrak{p}} - 1\big)\Bigg)(p-1) \\ &+ \mathtt{N}\left(1 + pe_{\mathfrak{p}} + p^2 e_{\mathfrak{p}} + \dots + p^{f_{\mathfrak{p}}-1}e_{\mathfrak{p}} + p^{f_{\mathfrak{p}}}(e_{\mathfrak{p}}-1)\right)(p-1) .\end{aligned}$$

Recall that $\sum_{i=1}^{e_{\mathfrak{p}}} \big(2i - e_{\mathfrak{p}} - 1\big) = 0$, therefore, after some computations, the above equation becomes

$$(p^{f_{\mathfrak{p}}}-1)x_{e_{\mathfrak{p}}f_{\mathfrak{p}}} = \big(p^{f_{\mathfrak{p}}}-1\big)\big(1-e_{\mathfrak{p}}\big)(p-1) + \mathtt{N}\left(p^{f_{\mathfrak{p}}}-1\right)\left(1-p+pe_{\mathfrak{p}}\right) ,$$

which cleary gives $x_{e_{\mathfrak{p}}f_{\mathfrak{p}}} = (1-e_{\mathfrak{p}})(p-1) + \mathtt{N}(1-p+pe_{\mathfrak{p}})$. Now one can use the explicit form of $W_{\mathfrak{p}}$ to obtain the full vector x.
Now, for any $\mathfrak{c} \in \mathfrak{C}$, and for any $\tau \in \Sigma$, the q-expansion of the partial Hasse invariant h_τ at $\infty(\mathfrak{c})$ is 1, by [DDW19, Lemma 1.4]. Therefore, any product of partial Hasse invariants will still have q-expansion at the cusp $\infty(\mathfrak{c})$ equal to 1. □

Lemma 4.2.3. *Let $\mathfrak{q} \subset \mathcal{O}_F$ be a prime ideal not dividing $p\mathfrak{n}$. For any paritious weight $(k, \mathsf{w}) \in \mathbb{Z}^\Sigma \times \mathbb{Z}$, and any form $f \in \mathcal{S}_{k,\mathsf{w}}(\mathfrak{n}; \mathbb{F})$, one has that*

$$h_{\mathrm{ex}}\big(T_{\mathfrak{q}}f\big) = T_{\mathfrak{q}}\big(h_{\mathrm{ex}}f\big) .$$

Proof. We will verify this on geometric q-expansion using the explicit description of the action of Hecke operators given by Equation 3.18. Recall that the Hasse invariant h_{ex} has q-expansion equal to 1 at all cusps $\infty(\mathfrak{c})$ (by [DDW19, Lemma 1.4]), therefore if $f_{\mathfrak{c}} = \sum_{\xi \in \mathfrak{c}_+} a_\xi q^\xi$ for $\mathfrak{c} \in \mathfrak{C}$, then $\big(h_{\mathrm{ex}}f\big)_{\mathfrak{c}} = \sum_{\xi \in \mathfrak{c}_+} a_\xi q^\xi$. Moreover, since h_{ex} has paritious weight 0, $h_{\mathrm{ex}}f$ is still paritious of weight w. Let $\mathfrak{q} \subset \mathcal{O}_F$ be a prime ideal not dividing $p\mathfrak{n}$, and let $\mathfrak{c}', \mathfrak{c}'' \in \mathfrak{C}$ and $\alpha, \beta \in F_+$ such that $\mathfrak{c}\mathfrak{q} = \alpha\mathfrak{c}'$ and $\mathfrak{c}\mathfrak{q}^{-1} = \beta\mathfrak{c}''$. Then by Equation 3.18

$$\begin{aligned} a^\circ_\xi\Big(\big(T_\mathfrak{q}(h_{\mathrm{ex}}f)\big)_\mathfrak{c}\Big) &= \mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1}\alpha^{-(\mathsf{wt}-k)/2+\kappa_{\mathrm{ex}}/2}a^\circ_{\alpha^{-1}\xi}\big((\langle\mathfrak{q}\rangle_\mathsf{w} h_{\mathrm{ex}}f)_{\mathfrak{c}'}\big) \\ &\quad + \beta^{-(\mathsf{wt}-k)/2+\kappa_{\mathrm{ex}}/2}a^\circ_{\beta^{-1}\xi}\big((h_{\mathrm{ex}}f)_{\mathfrak{c}''}\big)\ . \end{aligned}$$

Now let us look more closely at $\alpha^{\kappa_{\mathrm{ex}}} = \alpha^{(p-1)(\mathtt{Nt}+\mathrm{ex})}$. The same arguments will apply to β. As already seen before

$$\begin{aligned} \alpha^{\mathrm{ex}} &= \prod_{\mathfrak{p}|p}\prod_{j=1}^{f_\mathfrak{p}}\prod_{i=1}^{e_\mathfrak{p}}\tau^{(i)}_{\mathfrak{p},j}(\alpha)^{2(2i-e_\mathfrak{p}-1)} \\ &= \prod_{\mathfrak{p}|p}\prod_{j=1}^{f_\mathfrak{p}}\tau^{(1)}_{\mathfrak{p},j}(\alpha)^{\sum_{i=1}^{e_\mathfrak{p}}2(2i-e_\mathfrak{p}-1)} \quad, \text{because } \tau^{(i)}_{\mathfrak{p},j} \equiv \tau^{i+1}_{\mathfrak{p},j} \mod \varpi, \\ &= 1 \quad, \text{because } \sum_{i=1}^{e_\mathfrak{p}}2(2i-e_\mathfrak{p}-1) = 0. \end{aligned}$$

Moreover $\alpha^{(p-1)\mathtt{Nt}} = \mathrm{Nm}_{F/\mathbb{Q}}(\alpha)^{(p-1)\mathtt{N}} \equiv 1 \mod \varpi$, since $v_p(\alpha) = 0$. Therefore one has that:

$$\begin{aligned} a^\circ_\xi\Big(\big(T_\mathfrak{q}(h_{\mathrm{ex}}f)\big)_\mathfrak{c}\Big) &= \mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1}\alpha^{-\ell}a^\circ_{\alpha^{-1}\xi}\big((\langle\mathfrak{q}\rangle_\mathsf{w} f)_{\mathfrak{c}'}\big) + \beta^{-\ell}a^\circ_{\beta^{-1}\xi}\big(f_{\mathfrak{c}''}\big) \\ &= a^\circ_\xi\big((T_\mathfrak{q}f)_\mathfrak{c}\big) \\ &= a^\circ_\xi\big((h_{\mathrm{ex}}T_\mathfrak{q}f)_\mathfrak{c}\big)\ . \end{aligned}$$

□

4.2.2 Working in $R_m = \mathcal{O}/\varpi^m\mathcal{O}$

In order to work over rings $R_m = \mathcal{O}/\varpi^m\mathcal{O}$, one has to lift partial Hasse invariants and construct a product of partial Hasse invariants living in liftable weight with coefficients over R_m.

Liftings of h_τ^2 modulo ϖ^m

We recall here how Reduzzi-Xiao (see [RX17, Section 3.13.1]), via a method of Emerton, Reduzzi and Xiao (see [ERX17b, Section 3.3.1] construct lifts of the generalized partial Hasse invariants. Recall that we denote by w^τ the weight of the partial Hasse invariant h_τ, for $\tau \in \Sigma$ (see Definition 4.2.1).

Lemma 4.2.4. *Let M be a positive integer divisible by $2p^{m-1}$. For any $\tau \in \Sigma$, there exists a Hilbert modular form*

$$\tilde{h}_{\tau,M} \in \mathrm{H}^0\left(\mathrm{Sh}_{R_m}, \omega_{R_m}^{(Mw^\tau,0)}\right),$$

which is locally the $\frac{M}{2}$th power of a lift of $h_\tau^2 \in \mathrm{H}^0(\mathrm{Sh}_\mathbb{F}, \omega_\mathbb{F}^{(2w^\tau,0)})$.

Proof. Let $\mathcal{U}$ be an open affine covering of $\mathrm{Sh}^{\mathrm{tor}}$. Let $U \in \mathcal{U}$ be an open affine subscheme of $\mathrm{Sh}^{\mathrm{tor}}$ and let $h^2_{\tau,U_\mathbb{F}} \in \mathrm{H}^0(U_\mathbb{F}, \omega_\mathbb{F}^{(2w^\tau,0)})$ denote the restriction of the square of the generalized partial Hasse invariant at τ to $U_\mathbb{F} = U \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}\,\mathbb{F}$, for any $\tau \in \Sigma$. The form $h^2_{\tau,U_\mathbb{F}}$ can be lifted arbitrarily to an element $\tilde{h}_{\tau,U_{R_m}} \in \mathrm{H}^0(U_{R_m}, \omega_{R_m}^{(2w^\tau,0)})$, where $U_{R_m} = U \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(R_m)$. Since M is a positive integer divisible by p^{m-1}, the lift $\tilde{h}^M_{\tau,U_{R_m}}$ is independent of this arbitrary choice. We now deduce that the sections $\{\tilde{h}^M_{\tau,U_{R_m}}\}_{U\in\mathcal{U}}$ glue together into a global section

$$\tilde{h}_{\tau,M} \in \mathrm{H}^0(\mathrm{Sh}_{R_m}, \omega_{R_m}^{(Mw^\tau,0)}) ,$$

which is independent of the choice of affine covering $\mathcal{U}$ of $\mathrm{Sh}^{\mathrm{tor}}$ and locally is the $\frac{M}{2}$th power of a lift of $h_\tau^2 \in \mathrm{H}^0(\mathrm{Sh}_{\mathbb{F}}, \omega_{\mathbb{F}}^{(2w^\tau,0)})$. □

Lemma 4.2.5. *For any integer $m > 1$, let $\kappa_m = p^{m-1}(p-1)(\mathtt{Nt} + \mathrm{ex})$. Then for any integer $m > 1$, there exists a Hilbert modular form $h_{\mathrm{ex},m} \in \mathrm{H}^0\left(\mathrm{Sh}_{R_m}, \omega_{R_m}^{(\kappa_m,0)}\right)$, which locally is the p^{m-1}-th power of a lift of h_{ex}. Moreover, for any $\mathfrak{c} \in \mathfrak{C}$, the q-expansion of $h_{\mathrm{ex},m}$ at the cusp $\infty(\mathfrak{c})$ is 1.*

Proof. By Lemma 4.2.4, we know that for any $\tau \in \Sigma$, we can construct a Hilbert modular form $\tilde{h}_{\tau,2p^{m-1}} \in \mathrm{H}^0(\mathrm{Sh}_{R_m}, \omega_{R_m}^{(2p^{m-1}w^\tau,0)})$, which is locally the p^{m-1}-th power of h_τ^2. In particular, it will then have q-expansion equal to 1 at any cusp $\infty(\mathfrak{c})$ for any $\mathfrak{c} \in \mathfrak{C}$, since h_τ^2 has q-expansion equal to 1 at any cusp $\infty(\mathfrak{c})$, by [DDW19, Lemma 1.4]. Moreover, by Lemma 4.2.2, we know that there exists a product of generalized partial Hasse invariants $h_{\mathrm{ex}} \in \mathrm{H}^0\left(\mathrm{Sh}_{\mathbb{F}}, \omega_{\mathbb{F}}^{(\kappa_{\mathrm{ex}},0)}\right)$, with $\kappa_{\mathrm{ex}} = (p-1)(\mathtt{Nt} + \mathrm{ex})$. Let us write the product $h_{\mathrm{ex}} = \prod_{\tau\in\Sigma} h_\tau^{2c_\tau}$, with $c_\tau \in \mathbb{Z}$. Then taking $h_{\mathrm{ex},\,m}$ to be $\prod_{\tau\in\Sigma} \tilde{h}^{c_\tau}_{\tau,2p^{m-1}}$, which is an element of $\mathrm{H}^0\left(\mathrm{Sh}_{R_m}, \omega_{R_m}^{((p-1)p^{m-1}(\mathtt{Nt}+\mathrm{ex}),0)}\right)$, gives the result. □

Lemma 4.2.6. *Let $\mathfrak{q} \subset \mathcal{O}_F$ be a prime ideal not dividing $p\mathfrak{n}$, and let m be an integer $m > 1$. For any paritious weight $(k, \mathsf{w}) \in \mathbb{Z}^\Sigma \times \mathbb{Z}$, and any form $f \in \mathcal{S}_{k,\mathsf{w}}(\mathfrak{n}; R_m)$, one has that*

$$h_{\mathrm{ex},m}\big(T_\mathfrak{q} f\big) = T_\mathfrak{q}\big(h_{\mathrm{ex},m} f\big) .$$

Proof. We will verify this on geometric q-expansion using the explicit description of the action of Hecke operators given by Equation 3.18. Recall that the Hasse invariant $h_{\mathrm{ex},m}$ has q-expansion equal to 1 at all cusps $\infty(\mathfrak{c})$, therefore if $f_\mathfrak{c} = \sum_{\xi\in\mathfrak{c}_+\cup\{0\}} a_\xi q^\xi$ for $\mathfrak{c} \in \mathfrak{C}$, then $\big(h_{\mathrm{ex},m} f\big)_\mathfrak{c} = \sum_{\xi\in\mathfrak{c}_+\cup\{0\}} a_\xi q^\xi$. Moreover, since $h_{\mathrm{ex},m}$ has paritious weight 0, $h_{\mathrm{ex},m} f$ is still paritious of weight w. Let $\mathfrak{q} \subset \mathcal{O}_F$ be a prime ideal not dividing $p\mathfrak{n}$, and let $\mathfrak{c}', \mathfrak{c}'' \in \mathfrak{C}$ and $\alpha, \beta \in F_+$ such that $\mathfrak{c}\mathfrak{q} = \alpha\mathfrak{c}'$ and $\mathfrak{c}\mathfrak{q}^{-1} = \beta\mathfrak{c}''$. Then by Equation 3.18

$$\begin{aligned} a^\circ_\xi\Big(\big(T_\mathfrak{q}(h_{\mathrm{ex},m} f)\big)_\mathfrak{c}\Big) &= \mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1} \alpha^{-(\mathsf{w}t-k)/2+\kappa_m/2} a^\circ_{\alpha^{-1}\xi}\big((\langle\mathfrak{q}\rangle_\mathsf{w} h_{\mathrm{ex},m} f)_{\mathfrak{c}'}\big) \\ &\quad + \beta^{-(\mathsf{w}t-k)/2+\kappa_m/2} a^\circ_{\beta^{-1}\xi}\big((h_{\mathrm{ex},m} f)_{\mathfrak{c}''}\big) . \end{aligned}$$

Now let us look more closely at $\alpha^{\kappa_m} = \alpha^{p^{m-1}(p-1)(\mathtt{Nt}+\mathrm{ex})}$. The same arguments will apply to β. As already seen in the proof of Lemma 4.2.3, since $v_\mathfrak{p}(\alpha) = 0$, $\alpha^{(p-1)(\mathtt{Nt}+\mathrm{ex})} \equiv 1 \mod \varpi$ and therefore $\alpha^{p^{m-1}(p-1)(\mathtt{Nt}+\mathrm{ex})} \equiv 1 \mod \varpi^m$. The above equation then becomes

$$\begin{aligned} a^\circ_\xi\Big(\big(T_\mathfrak{q}(h_{\mathrm{ex},m} f)\big)_\mathfrak{c}\Big) &= \mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1} \alpha^{-(\mathsf{w}t-k)/2} a^\circ_{\alpha^{-1}\xi}\big((\langle\mathfrak{q}\rangle_\mathsf{w} f)_{\mathfrak{c}'}\big) + \beta^{-(\mathsf{w}t-k)/2} a^\circ_{\beta^{-1}\xi}\big(f_{\mathfrak{c}''}\big) \\ &= a^\circ_\xi\big((T_\mathfrak{q} f)_\mathfrak{c}\big) \\ &= a^\circ_\xi\big((h_{\mathrm{ex}} T_\mathfrak{q} f)_\mathfrak{c}\big) . \end{aligned}$$

□

4.3 Unramifiedness modulo ϖ

Here we proceed to prove Theorem 4.0.2. The existence of the representation $\rho_f : G_F \to \mathrm{GL}_2(\mathbb{F})$ of Theorem 4.0.2 follows from a standard argument, which is presented in the following Proposition. The difficulty of the proof of Theorem 4.0.2 lies in showing that the representation ρ_f is unramified at $\mathfrak{p}$.

Proposition 4.3.1. *Let $\mathfrak{p}$ be a fixed prime of F above p and let $\mathrm{S} = \mathrm{Supp}(p\mathfrak{n})$. Let $(k, 1)$ be a paritious weight such that $k_\tau = 1$ for all $\tau \in \Sigma_\mathfrak{p}$. Let $f \in \mathcal{S}_{k,1}(\mathfrak{n}, \mathbb{F})$ be an eigenform for all Hecke operators $T_\mathfrak{q}$, with eigenvalues $\lambda(f, \mathfrak{q})$, and for diamond operators $\langle \mathfrak{q} \rangle_1$, with eigenvalues $\epsilon(\mathfrak{q})$, for $\mathfrak{q}$ a prime ideal of $\mathcal{O}_F$, $\mathfrak{q} \notin \mathrm{S}$. Then there exists a semi-simple Galois representation $\rho_f : G_F \to \mathrm{GL}_2(\mathbb{F})$, which is unramified at all $\mathfrak{q} \notin \mathrm{S}$ and such that $\mathrm{tr}(\rho_f(\mathrm{Frob}_\mathfrak{q})) = \lambda(f, \mathfrak{q})$ and $\det(\rho_f(\mathrm{Frob}_\mathfrak{q})) = \epsilon(\mathfrak{q})\,\mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1}$, for all $\mathfrak{q} \notin \mathrm{S}$.*

Proof. By Lemma 4.1.6 there exists an integer r_0 such that for all $r \geq r_0$, one can lift paritious weight cuspidal Hilbert modular forms modulo p in characteristic 0, i.e.

$$\mathcal{S}_{k+r\cdot(\mathtt{Nt}+\mathrm{ex}),\mathsf{w}}(\mathfrak{n}; \mathcal{O}) \otimes \mathbb{F} \simeq \mathcal{S}_{k+r\cdot(\mathtt{Nt}+\mathrm{ex}),\mathsf{w}}(\mathfrak{n}; \mathbb{F}) \ .$$

Moreover, by Lemma 4.2.3, one knows that for any integer r the form $h_{\mathrm{ex}}^r f$ will still be an eigenform for all Hecke operators $T_\mathfrak{q}$, for $\mathfrak{q}$ as in the hypothesis, with same eigenvalues as f. Now it suffices to lift $h_{\mathrm{ex}}^r f$ for $r \geq r_0$ and to apply a theorem of Deligne-Serre [DS74, Lemme 6.11] to obtain a Galois representation $\rho : G_F \to \mathrm{GL}_2(\mathcal{O})$ such that $\mathrm{tr}(\rho(\mathrm{Frob}_\mathfrak{q})) \equiv \lambda(f, \mathfrak{q}) \mod \varpi$, for all $\mathfrak{q} \notin \mathrm{S}$. Therefore we take ρ_f to be the semi-simplification of the reduction modulo ϖ of the representation ρ given by Deligne-Serre. Moreover, the obtained representation is such that $\det(\rho_f(\mathrm{Frob}_\mathfrak{q}))$ corresponds to the eigenvalue of the operator $\langle \mathfrak{q} \rangle_\mathsf{w}\,\mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1}$. □

In order to show that ρ_f is unramified at $\mathfrak{p}$, we will apply the doubling method, see [Wie14]. We will therefore need two ways to go in higher weight. One is given by multiplying by the Hasse invariant h_{ex} and the second one is given by the the Frobenius Operator.

4.3.1 Frobenius Operator

Recall that $\mathfrak{p}$ is a fixed prime of $\mathcal{O}_F$ dividing p, we take a moment to recall here the action of the normalized Hecke operator $T_\mathfrak{p}^\circ$ on q-expansions, in the paritious weight (k, w) setting. Let $(k, \mathsf{w}) \in \mathbb{Z}^\Sigma \times \mathbb{Z}$ be a paritious weight. Let $f \in \mathcal{S}_{k,\mathsf{w}}(\mathfrak{n}; \mathcal{O})$ and let $f = (f_\mathfrak{c})_{\mathfrak{c} \in \mathfrak{C}}$, where $f_\mathfrak{c} = \sum_{\xi \in \mathfrak{c}_+} a_\xi q^\xi$ is its geometric q-expansion at the cusp $\infty(\mathfrak{c})$. Recall that we denote by $a_\xi^\circ(f_\mathfrak{c}) = \mathrm{Nm}(\mathfrak{c})^{-\mathsf{w}} a_\xi(f_\mathfrak{c})$ the normalized geometric coefficients. Let $\alpha, \beta \in F_+$ be such that $\mathfrak{c}\mathfrak{p} = \alpha \mathfrak{c}'$ and $\mathfrak{c}\mathfrak{p}^{-1} = \beta \mathfrak{c}''$, for $\mathfrak{c}, \mathfrak{c}', \mathfrak{c}'' \in \mathfrak{C}$. Then for any $\xi \in \mathfrak{c}_+$, Equation 3.18 can be written as

$$a_\xi^\circ\left((T_\mathfrak{p}^\circ f)_\mathfrak{c}\right) = \mathrm{Nm}(\mathfrak{p})^{\mathsf{w}-1} \varpi_\mathfrak{p}^{(k_\mathfrak{p}-\mathsf{w}t_\mathfrak{p})/2} \alpha^{(k-\mathsf{w}t)/2} a_{\alpha^{-1}\xi}^\circ\left((\langle \mathfrak{p} \rangle_\mathsf{w} f)_{\mathfrak{c}'}\right) + \varpi_\mathfrak{p}^{(k_\mathfrak{p}-\mathsf{w}t_\mathfrak{p})/2} \beta^{(k-\mathsf{w}t)/2} a_{\beta^{-1}\xi}^\circ\left(f_{\mathfrak{c}''}\right) . \tag{4.3}$$

Since we will be working between two sets of paritious weights, in what follows we will add the weights to the notation of the Hecke operator, i.e. will write $T_\mathfrak{p}^{\circ,(k,\mathsf{w})}$ for the normalized Hecke operator $T_\mathfrak{p}^\circ$ acting on $\mathcal{M}_{k,\mathsf{w}}(\mathfrak{n}; R)$.

Definition 4.3.2. Let $(k, 1) \in \mathbb{Z}^\Sigma \times \mathbb{Z}$ be a paritious weight such that $k_\tau = 1$ for all $\tau \in \Sigma_\mathfrak{p}$. We define the *Frobenius operator* at $\mathfrak{p}$ on modulo p modular forms

$$V_\mathfrak{p} : \mathcal{S}_{(k,1)}(\mathfrak{n}; \mathbb{F}) \longrightarrow \mathcal{S}_{k+\kappa_{\mathrm{ex}},1}(\mathfrak{n}; \mathbb{F})$$

to be

$$V_{\mathfrak{p}}(f) := \langle \mathfrak{p} \rangle_1^{-1} \big(h_{\mathrm{ex}} \cdot (T_{\mathfrak{p}}^{\circ,(k,1)} f) - T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)} (h_{\mathrm{ex}} \cdot f) \big).^{1}$$

We now proceed to calculate how the Frobenius operator acts on geometric q-expansions.

Proposition 4.3.3. *Let $(k,1) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}$ be such that $k_\tau = 1$ for all $\tau \in \Sigma_{\mathfrak{p}}$. Let $f \in \mathcal{S}_{k,1}(\mathfrak{n}, \mathbb{F})$ with $f = (f_{\mathfrak{c}})_{\mathfrak{c} \in \mathfrak{C}}$, where $f_{\mathfrak{c}} = \sum_{\xi \in \mathfrak{c}_+} a_\xi q^\xi$ is its geometric q-expansion at the cusp $\infty(\mathfrak{c})$. Then for any $\xi \in \mathfrak{c}_+$*

$$a_\xi^\circ \big((V_{\mathfrak{p}} f)_{\mathfrak{c}} \big) = \alpha^{-(\mathtt{t}-k)/2} a^\circ_{\alpha^{-1}\xi}(f_{\mathfrak{c}'}) \ ,$$

where $\alpha \in F_+$ and $\mathfrak{c}' \in \mathfrak{C}$ such that $\alpha \mathfrak{c}' = \mathfrak{c}\mathfrak{p}$, and $a_{\alpha^{-1}\xi} = 0$ if $\alpha^{-1}\xi \notin \mathfrak{c}'$.

Proof. Let us recall that the q-expansion of $h_{\mathrm{ex}} f$ and f are the same. Now since $k_\tau + (p-1)(N + \mathrm{ex}_\tau) > 1$ and $k_\tau = 1$ for all $\tau \in \Sigma_{\mathfrak{p}}$, by Equation 3.20, one has that for $\xi \in \mathfrak{c}_+$

$$\begin{aligned} a_\xi^\circ \big((T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)} (h_{\mathrm{ex}} f))_{\mathfrak{c}} \big) &= \varpi_{\mathfrak{p}}^{(p-1)(\mathtt{N}\mathtt{t}_{\mathfrak{p}} + \mathrm{ex}_{\mathfrak{p}})/2} \beta^{-(\mathtt{wt}-k)/2 + \kappa_{\mathrm{ex}}/2} a^\circ_{\beta^{-1}\xi} \big((h_{\mathrm{ex}} f)_{\mathfrak{c}''} \big) \\ &= \varpi_{\mathfrak{p}}^{(p-1)(\mathtt{N}\mathtt{t}_{\mathfrak{p}} + \mathrm{ex}_{\mathfrak{p}})/2} \beta^{\kappa_{\mathrm{ex}}/2} \beta^{-(\mathtt{wt}-k)/2} a^\circ_{\beta^{-1}\xi}(f_{\mathfrak{c}''}) \\ &= \beta^{-(\mathtt{wt}-k)/2} a^\circ_{\beta^{-1}\xi}(f_{\mathfrak{c}''}). \end{aligned}$$

where $\mathfrak{c}'' \in \mathfrak{C}'$ and $\beta \in F_+$ are such that $\beta \mathfrak{c}'' = \mathfrak{c}\mathfrak{p}^{-1}$. The last equality is given by the fact that $\varpi_{\mathfrak{p}}^{(p-1)(\mathtt{N}\mathtt{t}_{\mathfrak{p}} + \mathrm{ex}_{\mathfrak{p}})/2} \beta^{(p-1)(\mathtt{N}\mathtt{t} + \mathrm{ex})/2} = 1$ in $\mathbb{F}$. In fact, we already established that $\beta^{\mathrm{ex}} = 1$ in $\mathbb{F}$ (see for example proof of Lemma 4.2.3) and the same holds for $\varpi_{\mathfrak{p}}^{\mathrm{ex}_{\mathfrak{p}}}$. It suffices then to show that $\varpi_{\mathfrak{p}}^{(p-1)\mathtt{N}\mathtt{t}_{\mathfrak{p}}/2} \beta^{(p-1)\mathtt{N}\mathtt{t}/2} = 1$ in $\mathbb{F}$. Let us first remark that

$$\varpi_{\mathfrak{p}}^{\mathtt{t}_{\mathfrak{p}}} \operatorname{Nm}(\mathfrak{p})^{-1} \in \mathbb{Z}_p^\times$$

In fact, $v_{\mathfrak{p}}(\operatorname{Nm}(\mathfrak{p})) = v_{\mathfrak{p}}(p^{f_{\mathfrak{p}}}) = e_{\mathfrak{p}} f_{\mathfrak{p}} = \#\Sigma_{\mathfrak{p}}$. Therefore, since we are now working with parallel weights $(p-1)\mathtt{N}\mathtt{t}_{\mathfrak{p}}/2$ and $(p-1)\mathtt{N}\mathtt{t}/2$, one has that

$$\begin{aligned} \varpi_{\mathfrak{p}}^{(p-1)\mathtt{N}\mathtt{t}_{\mathfrak{p}}/2} \beta^{(p-1)\mathtt{N}\mathtt{t}/2} &= \varpi_{\mathfrak{p}}^{(p-1)\mathtt{N}\mathtt{t}_{\mathfrak{p}}/2} \operatorname{Nm}(\beta)^{(p-1)\mathtt{N}/2} \\ &= \varpi_{\mathfrak{p}}^{(p-1)\mathtt{N}\mathtt{t}_{\mathfrak{p}}/2} \operatorname{Nm}(\mathfrak{p})^{-(p-1)\mathtt{N}/2} \left(\frac{\operatorname{Nm}(\mathfrak{c})}{\operatorname{Nm}(\mathfrak{c}'')} \right)^{(p-1)\mathtt{N}/2} \quad \in \mathbb{Z}_p^\times \end{aligned}$$

which is congruent to 1 modulo ϖ, by Fermat's little theorem.
Let us now look at the q-expansion of $T_{\mathfrak{p}}^\circ f$ in $\mathcal{S}_{(k,1)}(\mathfrak{n}; \mathbb{F})$. Since $k_\tau = 1$ for all $\tau \in \Sigma_{\mathfrak{p}}$, for any $\xi \in \mathfrak{c}_+$ one has by Equation 4.3

$$a_\xi^\circ \big((h_{\mathrm{ex}} (T_{\mathfrak{p}}^{\circ,(k,1)} f))_{\mathfrak{c}} \big) = a_\xi^\circ \big((T_{\mathfrak{p}}^{\circ,(k,1)} f)_{\mathfrak{c}} \big) = \alpha^{-(\mathtt{wt}-k)/2} a^\circ_{\alpha^{-1}\xi} \big((\langle \mathfrak{p} \rangle_1 f)_{\mathfrak{c}'} \big) + \beta^{-(\mathtt{wt}-k)/2} a^\circ_{\beta^{-1}\xi}(f_{\mathfrak{c}''}),$$

where $\mathfrak{c}' \in \mathfrak{C}'$ and $\alpha \in F_+$ are such that $\alpha \mathfrak{c}' = \mathfrak{c}\mathfrak{p}$, and $a_{\alpha^{-1}\xi} = 0$ if $\alpha^{-1}\xi \notin \mathfrak{c}'$.
Combining the two formulae, one has that for any $\xi \in \mathfrak{c}_+$,

$$a_\xi^\circ \big((h_{\mathrm{ex}} (T_{\mathfrak{p}}^{\circ,(k,1)} f) - T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)} (h_{\mathrm{ex}} f))_{\mathfrak{c}} \big) = \alpha^{-(\mathtt{wt}-k)/2} a^\circ_{\alpha^{-1}\xi} \big((\langle \mathfrak{p} \rangle_1 f)_{\mathfrak{c}'} \big)$$

and therefore $a_\xi^\circ \big((V_{\mathfrak{p}} f)_{\mathfrak{c}} \big) = \alpha^{-(\mathtt{wt}-k)/2} a^\circ_{\alpha^{-1}\xi}(f_{\mathfrak{c}'})$. ∎

[1] Recall that we have defined normalized diamond operators in Section 3.1.

Remark 4.3.4. The adelic q-expansion for the action of the Frobenius operator makes sense only for parallel weights (see Remark 3.3.12), since we are working modulo ϖ. Therefore the HMF f has weights $\mathsf{w} = 1$, $k_\tau = 1$ for all $\tau \in \Sigma$, and the action of the Frobenius operator is then:

$$C(\mathfrak{m}, V_\mathfrak{p} f) = \mathrm{Nm}(\mathfrak{c})^{-1} a_\xi\big((V_\mathfrak{p} f)_\mathfrak{c}\big) = \mathrm{Nm}(\mathfrak{c}')^{-1} a_{\alpha^{-1}\xi}(f_\mathfrak{c}) = C(\mathfrak{m}\mathfrak{p}^{-1}, f) \ ,$$

for $\mathfrak{m}$ an integral ideal of $\mathcal{O}_F$, and $\mathfrak{m} = \xi\mathfrak{c}^{-1}$ for unique $\mathfrak{c} \in \mathfrak{C}$ and $\xi \in \mathfrak{c}_+$. This is the same formula of [DW18, Proposition 3.6] or as in [DDW19].

Proposition 4.3.5. *Let $(k, 1) \in \mathbb{Z}^\Sigma \times \mathbb{Z}$ be such that $k_\tau = 1$ for all $\tau \in \Sigma_\mathfrak{p}$. Let $f \in \mathcal{S}_{k,1}(\mathfrak{n}; \mathbb{F})$, then*

$$T_\mathfrak{p}^{\circ,(k+\kappa_{\mathrm{ex}},1)}\big(V_\mathfrak{p} f\big) = h_{\mathrm{ex}} \cdot f \ .$$

Proof. It suffices to check this on q-expansions. So let $\mathfrak{c} \in \mathfrak{C}$ and $f_\mathfrak{c} = \sum_{\xi \in \mathfrak{c}_+} a_\xi q^\xi$ be the q-expansion of f at the cusp $\infty(\mathfrak{c})$. Then for $\xi \in \mathfrak{c}_+$, as in the proof of Proposition 4.3.3, one has that

$$\begin{aligned} a_\xi^\circ\Big(\big(T_\mathfrak{p}^{\circ,(k+\kappa_{\mathrm{ex}},1)}(V_\mathfrak{p} f)\big)_\mathfrak{c}\Big) &= \varpi_\mathfrak{p}^{(p-1)(\mathtt{Nt}_\mathfrak{p}+\mathtt{ex}_\mathfrak{p})/2} \beta^{(p-1)(\mathtt{Nt}+\mathtt{ex})/2} \beta^{-(\mathsf{wt}-k)/2} a_{\beta^{-1}\xi}^\circ\big((V_\mathfrak{p} f)_{\mathfrak{c}''}\big) \\ &= \beta^{-(\mathsf{wt}-k)/2} a_{\beta^{-1}\xi}^\circ\big((V_\mathfrak{p} f)_{\mathfrak{c}''}\big) \ , \end{aligned}$$

where $\beta \in F_+$ and $\mathfrak{c}'' \in \mathfrak{C}$ are such that $\beta\mathfrak{c}'' = \mathfrak{c}\mathfrak{p}^{-1}$. Now we want to apply the previous Proposition to the cusp $\infty(\mathfrak{c}'')$, and since $\mathfrak{c}''\mathfrak{p} = \beta^{-1}\mathfrak{c}$, one has that

$$\begin{aligned} a_\xi^\circ\Big(\big(T_\mathfrak{p}^{\circ,(k+\kappa_{\mathrm{ex}},1)}(V_\mathfrak{p} f)\big)_\mathfrak{c}\Big) &= \beta^{-(\mathsf{wt}-k)/2}\left(\beta^{(\mathsf{w}-k)/2} a_\xi^\circ(f_\mathfrak{c})\right) \\ &= a_\xi^\circ(f_\mathfrak{c}) \\ &= a_\xi^\circ\big((h_{\mathrm{ex}} f)_\mathfrak{c}\big) \ . \end{aligned}$$

□

Proposition 4.3.6. *Let $(k, 1) \in \mathbb{Z}^\Sigma \times \mathbb{Z}$ be such that $k_\tau = 1$ for all $\tau \in \Sigma_\mathfrak{p}$. Let $f \in \mathcal{S}_{k,1}(\mathfrak{n}; \mathbb{F})$ be an eigenform for all $T_\mathfrak{q}$, for $\mathfrak{q}$ a prime of $\mathcal{O}_F$, $\mathfrak{q} \notin \mathrm{S}$, and suppose that it is also an eigenform for $T_\mathfrak{p}^{\circ,(k,1)}$ with eigenvalue $\lambda^\circ(f, \mathfrak{p})$.*

1. *The forms $h_{\mathrm{ex}} \cdot f$ and $V_\mathfrak{p} f$ are $\mathbb{F}$-linearly independent.*

2. *The $\mathbb{F}$-vector space $W := \mathbb{F}\big(h_{\mathrm{ex}} \cdot f\big) \oplus \mathbb{F}\big(V_\mathfrak{p} f\big)$ is stable under $T_\mathfrak{p}^{\circ,(k+\kappa_{\mathrm{ex}},1)}$, which acts via the matrix $\begin{pmatrix} T_\mathfrak{p}^{\circ,(k,1)} & 1 \\ -\langle\mathfrak{p}\rangle_1 & 0 \end{pmatrix}$ with respect to the basis $\{h_{\mathrm{ex}}, V_\mathfrak{p} f\}$. In particular, $T_\mathfrak{p}^{\circ,(k+\kappa_{\mathrm{ex}},1)}$ is invertible on W.*

3. *The Hecke operator $T_\mathfrak{q}$ in weight $(k + \kappa_{\mathrm{ex}}, 1)$ acts scalarly on W.*

Proof. 1. Suppose that there exists $\lambda \in \mathbb{F}^\times$ such that $V_\mathfrak{p} f = \lambda h_{\mathrm{ex}} f$. Let $\mathfrak{c} \in \mathfrak{C}$ and $\xi \in \mathfrak{c}_+$, then

$$a_\xi^\circ\big((V_\mathfrak{p} f)_\mathfrak{c}\big) = \lambda\, a_\xi^\circ\big((h_{\mathrm{ex}} f)_\mathfrak{c}\big) = \lambda\, a_\xi^\circ(f_\mathfrak{c}) \ .$$

By Proposition 4.3.3, the above equation becomes

$$\alpha^{-(\mathsf{wt}-k)/2} a_{\alpha^{-1}\xi}^\circ(f_{\mathfrak{c}'}) = \lambda\, a_\xi^\circ(f_\mathfrak{c}) \ ,$$

where $\mathfrak{c}' \in \mathfrak{C}$ and $\alpha \in F_+$ are such that $\alpha\mathfrak{c}' = \mathfrak{c}\mathfrak{p}$, and $a_{\alpha^{-1}\xi} = 0$ if $\alpha^{-1}\xi \notin \mathfrak{c}'$. Let us consider the set $\{n \in \mathbb{Z}_{>0} :$ there exists $\mathfrak{c} \in \mathfrak{C}$ et $\xi \in \mathfrak{c}_+$ such that $a^\circ_\xi(f_\mathfrak{c}) \neq 0$ et $v_\mathfrak{p}(\xi) = n\}$. This set is non-empty since $f \neq 0$, and therefore it admits a minimum, n_0. Let $\mathfrak{c}$ and $\xi \in \mathfrak{c}_+$ the elements realizing the minimum n_0. Then one has the following contradiction

$$0 \neq \lambda\, a^\circ_\xi(f_\mathfrak{c}) = \alpha^{-(\mathsf{wt}-k)/2} a^\circ_{\alpha^{-1}\xi}(f_{\mathfrak{c}'}) = 0\ .$$

In fact, since $v_\mathfrak{p}(\alpha) > 1$, $v_\mathfrak{p}(\alpha^{-1}\xi) < n_0$, which by minimality implies that $a_{\alpha^{-1}\xi}(f_{\mathfrak{c}'}) = 0$.

2. Let us recall that by Proposition 4.3.5, we already know that $T_\mathfrak{p}^{\circ,(k+\kappa_{\mathrm{ex}},1)}(V_\mathfrak{p} f) = h_{\mathrm{ex}} f \in W$. Now we will look at the action of $T_\mathfrak{p}^{(k+\kappa_{\mathrm{ex}},1)}$ on $h_{\mathrm{ex}} f$. By definition of $V_\mathfrak{p}$, we can write

$$\langle \mathfrak{p} \rangle_1 V_\mathfrak{p} f = h_{\mathrm{ex}}\big(T_\mathfrak{p}^{\circ,(k,1)} f\big) - T_\mathfrak{p}^{\circ,(k+\kappa_{\mathrm{ex}},1)}(h_{\mathrm{ex}} f),$$

which gives the desired result for the matrix $\begin{pmatrix} T_\mathfrak{p}^{\circ,(k,1)} & 1 \\ -\langle \mathfrak{p} \rangle_1 & 0 \end{pmatrix}$. This also means that $T_\mathfrak{p}^{\circ,(k+\kappa_{\mathrm{ex}},1)}$ is annihilated by $X^2 - T_\mathfrak{p}^{\circ,(k,1)} X + \langle \mathfrak{p} \rangle_1$ on W.

3. It follows directly from Lemma 4.2.3 and by commutativity of Hecke operators (see Proposition C.1.1 and Proposition C.1.3).

□

4.3.2 Nearly-ordinary Hilbert modular forms

The following definition is due to Hida (see [Hid89b]) (cf. also [Dim05, Definition 1.3]).

Definition 4.3.7. Let $f \in \mathcal{S}_{k,\mathsf{w}}(\mathfrak{n}; \mathcal{O})$ be a Hilbert modular eigenform. We suppose that $\mathcal{O}$ contains all the Hecke eigenvalues of f. We say that f is *nearly-ordinary at* $\mathfrak{p}$ if its $T^\circ_\mathfrak{p}$-eigenvalue is a p-adic unit, i.e. it lies in $\mathcal{O}^\times$.

We recall the reader that we work with arithmetic Frobenius elements, therefore we normalize the Artin recipocity map so that a uniformizer $\varpi_\mathfrak{p}$ is sent to an arithmetic Frobenius $\mathrm{Frob}_\mathfrak{p}$. Moreover, we take a *cyclotomic character* χ_{cyc} corresponding via global class field theory to the idele class character $\chi_{\mathrm{cyc}} : F_+^\times \backslash \mathbb{A}_{F,f}^\times \to \mathbb{Z}_p^\times$ sending y to $\prod_{\mathfrak{p}|p} \mathrm{Nm}^{-1}_{F_\mathfrak{p}/\mathbb{Q}_p}(y_\mathfrak{p})|y_f|_F^{-1}$. In particular, it is such that $\chi_{\mathrm{cyc}}(\varpi_\mathfrak{q}) = \mathrm{Nm}(\mathfrak{q})$ for $\mathfrak{q}$ not dividing p and $\chi_{\mathrm{cyc}}(\varpi_\mathfrak{p}) = \mathrm{Nm}(\mathfrak{p}) \prod_{\tau \in \Sigma_\mathfrak{p}} \tau(\varpi_\mathfrak{p})^{-1}$.[2] The following result is due results of Wiles ([Wil88, Theorem 2]) and Hida ([Hid89a, Theorem I.]) and by local-global compatibility for Hilbert modular forms by works of Saito ([Sai09]) and Skinner ([Ski09]).

Theorem 4.3.8 (Hida, Saito, Skinner, Wiles)**.** *Let (k, w) be a paritious weight such that $k_\tau > 1$ for all $\tau \in \Sigma_\mathfrak{p}$. Let $f \in \mathcal{S}_{k,\mathsf{w}}(\mathfrak{n}, \mathcal{O})$ be a nearly-ordinary at $\mathfrak{p}$ Hilbert modular eigenform with $T^\circ_\mathfrak{p}$-eigenvalue $\lambda(f, T^\circ_\mathfrak{p}) \in \mathcal{O}^\times$. Then the associated Galois representation $\rho_f : G_F \to \mathrm{GL}_2(\mathcal{O})$ is such that*

$$\rho_{f|D_\mathfrak{p}} \sim \begin{pmatrix} \chi_1 & \star \\ 0 & \chi_2 \end{pmatrix}$$

[2] Our choice of cyclotomic character is the inverse of the cyclotomic character used by Barrera, Dimitrov and Jorza (see [BDJ17, Notation]). This is because they work with geometric Frobenius and we want to work with arithmetic Frobenius.

where χ_1 and χ_2 are characters such that on the inertia they are respectively obtained by composing the Artin reciprocity map $I_{\mathfrak{p}} \to \mathcal{O}_{F,\mathfrak{p}}^\times$ with the maps

$$\hat{\chi}_1 : \mathcal{O}_{F,\mathfrak{p}}^\times \to \mathcal{O}^\times, \quad x \mapsto \prod_{\tau \in \Sigma_{\mathfrak{p}}} \tau(x)^{(k_\tau + \mathsf{w} - 2)/2};$$
$$\hat{\chi}_2 : \mathcal{O}_{F,\mathfrak{p}}^\times \to \mathcal{O}^\times, \quad x \mapsto \prod_{\tau \in \Sigma_{\mathfrak{p}}} \tau(x)^{(\mathsf{w} - k_\tau)/2}.$$

Moreover, $\hat{\chi}_2(\varpi_{\mathfrak{p}}) = \alpha_{\mathfrak{p}}^\circ$, the unit root of the normalized Hecke polynomial

$$X^2 - \lambda(f, T_{\mathfrak{p}}^\circ)X + \epsilon(\mathfrak{p})\chi_{\mathrm{cyc}}^{\mathsf{w}-1}(\varpi_{\mathfrak{p}}) \prod_{\tau \in \Sigma_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{k_\tau - 1}, \tag{4.4}$$

where $\epsilon(\mathfrak{p})$ is the eigenvalue of f for the operator $\langle \mathfrak{p} \rangle_{\mathsf{w}}$.

Remark 4.3.9. Let us remark that by Equation 4.4, $\alpha_{\mathfrak{p}}^\circ$ is congruent to $\lambda(f, T_{\mathfrak{p}}^\circ)$ modulo ϖ. This is due to the fact that we are supposing $k_\tau > 1$ for all $\tau \in \Sigma_{\mathfrak{p}}$.

Remark 4.3.10. We take a moment to discuss the Hecke polynomials. By our choice of convention and normalization, the representation ρ_f is such that the characteristic polynomial of $\rho_f(\mathrm{Frob}_{\mathfrak{q}})$ is given by:

$$X^2 - \lambda(f, T_{\mathfrak{q}})X + \epsilon(\mathfrak{q}) \,\mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1}, \tag{4.5}$$

where $\epsilon(\mathfrak{q})$ is the eigenvalue of $\langle \mathfrak{q} \rangle_{\mathsf{w}}$ for the eigenform f. Now, the Hecke operator $T_{\mathfrak{p}}^\circ$ is normalized and by abuse of notation we can see it as

$$T_{\mathfrak{p}}^\circ = \prod_{\tau \in \Sigma_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{(k_\tau - \mathsf{w})/2} T_{\mathfrak{p}},$$

where the operator $T_{\mathfrak{p}}$ is not well defined, but we use it here to highlight the different normalization taken for the Hecke operator at $\mathfrak{p}$ compared to the operator $T_{\mathfrak{q}}$. In particular, the Hecke polynomial of Equation 4.5 will not have integral roots for $\mathfrak{q} = \mathfrak{p}$. So the polynomial of Equation 4.4 is obtained by multiplying the roots the Hecke polynomial of Equation 4.5 by the factor $\prod_{\tau \in \Sigma_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{(k_\tau - \mathsf{w})/2}$.

Remark 4.3.11. Let $(k, 1)$ be a paritious weight such that $k_\tau = 1$ for all $\tau \in \Sigma_{\mathfrak{p}}$. We remark that for paritious weights $(k + r\kappa_{\mathrm{ex}}, 1)$, the characters χ_1 and χ_2 of Theorem 4.3.8 are unramified modulo ϖ. In fact, for $x \in \mathcal{O}_{F,\mathfrak{p}}^\times$, one has that

$$\begin{aligned}
\hat{\chi}_2(x) &= \prod_{\tau \in \Sigma_{\mathfrak{p}}} \tau(x)^{\frac{1-(k_\tau + r\kappa_{\mathrm{ex},\tau})}{2}} \text{ , since } \mathsf{w} = 1 \\
&= \prod_{\tau \in \Sigma_{\mathfrak{p}}} \tau(x)^{\frac{-r\kappa_{\mathrm{ex},\tau}}{2}} \text{ , since } k_\tau = 1 \text{ for all } \tau \in \Sigma_{\mathfrak{p}} \\
&= \prod_{j=1}^{f_{\mathfrak{p}}} \prod_{i=1}^{e_{\mathfrak{p}}} \tau_{\mathfrak{p},j}^{(i)}(x)^{-\frac{r(p-1)}{2}(\mathtt{N} + 2(2i - e_{\mathfrak{p}} - 1))} \text{ , (recall that } \mathtt{N} \text{ is even.)} \\
&\equiv \prod_{j=1}^{f_{\mathfrak{p}}} \tau_{\mathfrak{p},j}(x)^{-\frac{r(p-1)\mathtt{N}e_{\mathfrak{p}}}{2}} \mod \varpi \text{ , since } \tau_{\mathfrak{p},j}^{(i)} \equiv \tau_{\mathfrak{p},j}^{(i-1)} \mod \varpi \\
&\equiv \tau_{\mathfrak{p},1}(x)^{-\frac{r\mathtt{N}e_{\mathfrak{p}}}{2}(p-1)(1+p+\ldots+p^{f_{\mathfrak{p}}-1})} \mod \varpi \text{ , since } \tau_{\mathfrak{p},j} \equiv \tau_{\mathfrak{p},j-1}^{p} \mod \varpi \\
&\equiv 1 \mod \varpi.
\end{aligned}$$

The same calculations apply to $\hat{\chi}_1$. Therefore, the characters $\chi_1 \mod \varpi$ and $\chi_2 \mod \varpi$ are unramified at $\mathfrak{p}$.

4.3.3 Proof of Theorem 4.0.2

We will follow the strategy of the proof of [DW18, Theorem 1.1] to prove our result.

First, let us introduce a piece of notation. Let R be an $\mathcal{O}$-algebra and le ϵ denote an R-valued finite order Hecke character of F of conductor dividing $\mathfrak{n}$. We denote by $\mathcal{S}_{k,\mathsf{w}}(\mathfrak{n},\epsilon;R)$ the sub-R-module of $\mathcal{S}_{k,\mathsf{w}}(\mathfrak{n};R)$ of forms on which the operators $\langle\mathfrak{q}\rangle_{\mathsf{w}}$ act via $\epsilon(\mathfrak{q})$, for all prime ideals $\mathfrak{q}\subset\mathcal{O}_F$ coprime with $\mathfrak{n}$.

Now, let us recall that we are considering a cuspidal form $f\in\mathcal{S}_{(k,1)}(\mathfrak{n};\mathbb{F})$, which is a common eigenvector for all Hecke operators $T_{\mathfrak{q}}^{(k,1)}$ for all $\mathfrak{q}\notin \mathrm{S}\supset\{v: a \text{ a place of } F, v\neq\mathfrak{p} \text{ and } v|p\mathfrak{n}\}$. Since the diamond operators $\langle\mathfrak{q}\rangle_1$ commute with all Hecke operators, there exists an $\mathbb{F}^\times$-valued Hecke character ϵ, whose conductor divides $\mathfrak{n}$ and a form, still denoted $f\in\mathcal{S}_{(k,1)}(\mathfrak{n},\epsilon;\mathbb{F})$, sharing the same eigenvalues as f. So from now on, we will work with such an eigenform $f\in\mathcal{S}_{(k,1)}(\mathfrak{n},\epsilon;\mathbb{F})$. The following is a corollary of the previous theorem.

Corollary 4.3.12. *Let* $f\in\mathcal{S}_{(k,1)}(\mathfrak{n},\epsilon;\mathbb{F})$ *be an eigenvector for all Hecke operators* $T_{\mathfrak{q}}^{(k,1)}$, *for* $\mathfrak{q}\notin\mathrm{S}=\mathrm{supp}(p\mathfrak{n})$ *and for* $T_{\mathfrak{p}}^{\circ,(k,1)}$ *with eigenvalues* $\lambda(f,\mathfrak{q})$ *and* $\lambda^\circ(f,\mathfrak{p})$ *respectively. Let* $\alpha_{\mathfrak{p}}^\circ\in\mathbb{F}^\times$ *be a root of* $X^2-\lambda^\circ(f,\mathfrak{p})X+\epsilon(\mathfrak{p})$. *Then* $\rho_{f|D_{\mathfrak{p}}}: D_{\mathfrak{p}}\to\mathrm{GL}_2(\mathbb{F})$ *admits a* 1*-dimensional unramified quotient on which* $\mathrm{Frob}_{\mathfrak{p}}$ *acts by* $\alpha_{\mathfrak{p}}^\circ$.

Proof. Let us construct the subspace $W:=\mathbb{F}(h_{\mathrm{ex}}f)\oplus\mathbb{F}(V_{\mathfrak{p}}f)\subset\mathcal{S}_{k+\kappa_{\mathrm{ex}},1}(\mathfrak{n},\epsilon;\mathbb{F})$ as in Proposition 4.3.6. By part 2. of this Proposition, we know that $T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)}$ acts on this W via the matrix $\begin{pmatrix}\lambda^\circ(f,\mathfrak{p}) & 1\\ -\epsilon(\mathfrak{p}) & 0\end{pmatrix}$, and by hypothesis $\alpha_{\mathfrak{p}}^\circ$ is an eigenvalue of this matrix. Therefore there exists $f_{\alpha_{\mathfrak{p}}^\circ}\in W$ which is an eigenform for all Hecke operators $T_{\mathfrak{q}}^{(k+\kappa_{\mathrm{ex}},1)}$ with eigenvalues $\lambda(f,\mathfrak{q})$ for $\mathfrak{q}\nmid\mathfrak{n}p$ and for the Hecke operator $T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)}$ with eigenvalue $\alpha_{\mathfrak{p}}^\circ\in\mathbb{F}^\times$. Multiplying by a big enough power of h_{ex}, we can bring the system of eigenvalues of $f_{\alpha_{\mathfrak{p}}^\circ}$ to liftable weight and in particular, by Lemma 4.1.6, there exists $\widetilde{f}\in\mathcal{S}_{k+r\kappa_{\mathrm{ex}},1}(\mathfrak{n},\widetilde{\epsilon};\mathcal{O})$ with eigenvalues lifting those of $f_{\alpha_{\mathfrak{p}}^\circ}$, where $\widetilde{\epsilon}$ is a lift of ϵ. Moreover, $\widetilde{f}$ is nearly-ordinary at $\mathfrak{p}$, since $T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)}\widetilde{f}=\widetilde{\alpha_{\mathfrak{p}}^\circ}\widetilde{f}$, where $\widetilde{\alpha_{\mathfrak{p}}^\circ}\in\mathcal{O}^\times$ is a lift of $\alpha_{\mathfrak{p}}^\circ$, which is not 0 in $\mathbb{F}$. Now by Theorem 4.3.8, the Galois representation $\rho_{\tilde{f}}$ attached to $\tilde{f}$ is of the form

$$\rho_{f|D_{\mathfrak{p}}}\sim\begin{pmatrix}\chi_1 & \star\\ 0 & \chi_2\end{pmatrix}$$

where χ_1 and χ_2 are characters such that they are unramified modulo ϖ by Remark 4.3.11. Therefore, ρ_f admits an unramified quotient. Moreover, $\mathrm{Frob}_{\mathfrak{p}}$ acts on this unramified quotient via $\alpha_{\mathfrak{p}}^\circ$, since by Theorem 4.3.8 $\hat{\chi}_2(\varpi_{\mathfrak{p}})=\widetilde{\alpha_{\mathfrak{p}}^\circ}\equiv\alpha_{\mathfrak{p}}^\circ \mod \varpi$. □

Now one has to distinguish two cases: when the polynomial $X^2-\lambda^\circ(f,\mathfrak{p})X+\epsilon(\mathfrak{p})$ admits distinct roots $\alpha_{\mathfrak{p}}^\circ\neq\beta_{\mathfrak{p}}^\circ$ and when it has a double root $\alpha_{\mathfrak{p}}^\circ$.

Distinct roots $\alpha_{\mathfrak{p}}^\circ\neq\beta_{\mathfrak{p}}^\circ$

If $X^2-\lambda^\circ(f,\mathfrak{p})X+\epsilon(\mathfrak{p})$ has two distinct roots $\alpha_{\mathfrak{p}}^\circ$ and $\beta_{\mathfrak{p}}^\circ$, it suffices to apply Corollary 4.3.12 to $(f,\alpha_{\mathfrak{p}}^\circ)$ and to $(f,\beta_{\mathfrak{p}}^\circ)$ to get that $\rho_{f|D_{\mathfrak{p}}}$ admits two distinct unramified quotients on which $\mathrm{Frob}_{\mathfrak{p}}$

acts via $\alpha_{\mathfrak{p}}^\circ$ and $\beta_{\mathfrak{p}}^\circ$. Therefore ρ_f is unramified at $\mathfrak{p}$. Moreover, $\mathrm{tr}\left(\rho_f(\mathrm{Frob}_{\mathfrak{p}})\right) = \alpha_{\mathfrak{p}}^\circ + \beta_{\mathfrak{p}}^\circ = \lambda^\circ(f,\mathfrak{p})$. This proves the theorem in this case.

Double root $\alpha_{\mathfrak{p}}^\circ$

We now treat the case where $X^2 - \lambda^\circ(f,\mathfrak{p})X + \epsilon(\mathfrak{p}) = (X-\alpha_{\mathfrak{p}}^\circ)^2$, and we will need to introduce some notation. Let

$$\mathbf{T}^{(k,1)} := \mathrm{im}\left(\mathcal{O}[T_{\mathfrak{q}}^{(k,1)}, \langle\mathfrak{q}\rangle_1]_{\mathfrak{q}\nmid p\mathfrak{n}} \to \mathrm{End}_{\mathcal{O}}\left(\mathcal{S}_{k,1}(\mathfrak{n};\mathcal{O})\right)\right)$$

denote the Hecke algebra acting on $\mathcal{S}_{(k,1)}(\mathfrak{n};\mathcal{O})$ and set $\widetilde{\mathbf{T}}^{(k,1)} := \mathbf{T}^{(k,1)}[T_{\mathfrak{p}}^{\circ,(k,1)}]$ inside the ring $\mathrm{End}_{\mathcal{O}}\left(\mathcal{S}_{k,1}(\mathfrak{n};\mathcal{O})\right)$. Moreover, we will denote

$$\mathbf{T}_{\mathbb{F}}^{(k,1)} := \mathrm{im}\left(\mathcal{O}[T_{\mathfrak{q}}^{(k,1)}, \langle\mathfrak{q}\rangle_1]_{\mathfrak{q}\nmid p\mathfrak{n}} \to \mathrm{End}_{\mathcal{O}}\left(\mathcal{S}_{k,1}(\mathfrak{n};\mathbb{F})\right)\right)$$

and $\widetilde{\mathbf{T}}_{\mathbb{F}}^{(k,1)} := \mathbf{T}_{\mathbb{F}}^{(k,1)}[T_{\mathfrak{p}}^{\circ,(k,1)}]$ inside $\mathrm{End}_{\mathcal{O}}\left(\mathcal{S}_{k,1}(\mathfrak{n};\mathbb{F})\right)$. Recall that by Lemma 4.1.6, $(k+r\kappa_{\mathrm{ex}},1)$ is a liftable weight and therefore one has an isomorphism

$$\mathbf{T}_{\mathbb{F}}^{(k+r\kappa_{\mathrm{ex}},1)} \xrightarrow{\sim} \mathbf{T}^{(k+r\kappa_{\mathrm{ex}},1)} \otimes_{\mathcal{O}} \mathbb{F}\ ,$$

which induces a surjection $\mathbf{T}^{(k+r\kappa_{\mathrm{ex}},1)} \twoheadrightarrow \mathbf{T}_{\mathbb{F}}^{(k+r\kappa_{\mathrm{ex}},1)}$. Moreover, by Lemma 4.2.3, one has a surjection $\mathbf{T}_{\mathbb{F}}^{(k+r\kappa_{\mathrm{ex}},1)} \twoheadrightarrow \mathbf{T}_{\mathbb{F}}^{(k,1)}$. To recap:

$$\mathbf{T}^{(k+r\kappa_{\mathrm{ex}},1)} \twoheadrightarrow \mathbf{T}_{\mathbb{F}}^{(k+r\kappa_{\mathrm{ex}},1)} \twoheadrightarrow \mathbf{T}_{\mathbb{F}}^{(k,1)} \tag{4.6}$$

Let $\mathbf{m}$ denote the maximal ideal of $\mathbf{T}_{\mathbb{F}}^{(k,1)}$ corresponding to the Hilbert modular cuspform $f \in \mathcal{S}_{(k,1)}(\mathfrak{n};\mathbb{F})$ of Theorem 4.0.2. We will also denote by $\mathbf{m}$ the maximal ideal of $\mathbf{T}_{\mathbb{F}}^{(k+r\kappa_{\mathrm{ex}},1)}$ or of $\mathbf{T}^{(k+r\kappa_{\mathrm{ex}},1)}$ defined as the pullback of $\mathbf{m} \subset \mathbf{T}_{\mathbb{F}}^{(k,1)}$, via the surjections in Equation (4.6). Let $\widetilde{\mathbf{m}}$ denote the maximal ideal of $\widetilde{\mathbf{T}}_{\mathbb{F}}^{(k,1)}$ corresponding to $(f,\alpha_{\mathfrak{p}}^\circ)$. We will still denote $\widetilde{\mathbf{m}}$ the ideal of $\widetilde{\mathbf{T}}_{\mathbb{F}}^{(k+r\kappa_{\mathrm{ex}},1)}$ or of $\widetilde{\mathbf{T}}^{(k+r\kappa_{\mathrm{ex}},1)}$, defined as the pullback of $\widetilde{\mathbf{m}} \subset \widetilde{\mathbf{T}}_{\mathbb{F}}^{(k,1)}$.

Lemma 4.3.13. *The Hecke operator $T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)}$ does not belong to $\mathbf{T}_{\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)} \otimes_{\mathcal{O}} \mathbb{F}$. Moreover, it does not belong to $\mathbf{T}_{\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)}$.*

Proof. By the lifting Lemma 4.1.6, $\mathbf{T}_{\mathbb{F}}^{(k+r\kappa_{\mathrm{ex}},1)} \xrightarrow{\sim} \mathbf{T}^{(k+r\kappa_{\mathrm{ex}},1)} \otimes_{\mathcal{O}} \mathbb{F}$. So if $T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)}$ belonged to $\mathbf{T}_{\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)}$, it would belong to $\mathbf{T}_{\mathbb{F},\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)}$. Let us now proceed to show that this is impossible. Let $W \subset \mathcal{S}_{k+\kappa_{\mathrm{ex}},1}(\mathfrak{n};\mathbb{F})$ denote the $\mathbb{F}$-vector space of Proposition 4.3.6, and recall that on this space the Hecke operators $T_{\mathfrak{q}}^{(k+\kappa_{\mathrm{ex}},1)}$ act scalarly, while the operator $T_{\mathfrak{p}}^{((k+\kappa_{\mathrm{ex}},1))}$ acts with minimal polynomial $X^2 - \lambda^\circ(f,\mathfrak{p})X + \epsilon(\mathfrak{p}) = (X-\alpha_{\mathfrak{p}}^\circ)^2$. One has the inclusion

$$h_{\mathrm{ex}}^{r-1} W \subset \mathcal{S}_{k+r\kappa_{\mathrm{ex}},1}(\mathfrak{n};\mathbb{F})_{\mathbf{m}}\ ,$$

which is equivariant for all Hecke operators $T_{\mathfrak{q}}$, for $\mathfrak{q} \notin \mathrm{S}$. So if $T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)}$ belonged to $\mathbf{T}_{\mathbb{F},\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)}$, it would belong to the Hecke algebra generated by the operators $T_{\mathfrak{q}}^{(k+r\kappa_{\mathrm{ex}},1)}$ for $\mathfrak{q} \notin \mathrm{S}$ acting on $h_{\mathrm{ex}}^{r-1}W$. But by what we have said before, we know that this algebra acts via a character, while $T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)}$ does not act semi-simply. $\square$

We can now proceed to complete the proof of Theorem 4.0.2 for the case where $X^2 - \lambda^\circ(f,\mathfrak{p})X + \epsilon(\mathfrak{p}) = (X - \alpha_{\mathfrak{p}}^\circ)^2$. Since $\mathbf{T}_{\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)}$ is torsion free and generated by Hecke operators away from the level and p, one has that $\mathbf{T}_{\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)} \otimes_{\mathcal{O}} K \simeq \prod_{g\in\mathcal{N}} K$, where $\mathcal{N}$ denotes the set of all newforms contributing to $\mathcal{S}_{k+r\kappa_{\mathrm{ex}},1}(\mathfrak{n};\mathcal{O})_{\mathbf{m}}$, where we now suppose that $\mathcal{O}$ is big enough to contain all eigenvalues of all newforms $g \in \mathcal{N}$.

Recall that by Proposition 4.3.1 we already have constructed the Galois representation ρ_f attached to f, which is semi-simple. If ρ_f is not irreducible, then it is the sum of two characters $\chi_1 \oplus \chi_2$. In particular, since the determinant of ρ_f is unramified at $\mathfrak{p}$, then the product $\chi_1\chi_2$ is unramified at $\mathfrak{p}$. So if one of the two was ramified at $\mathfrak{p}$, so would the other, but this contradicts the existence of an unramified quotient of Corollary 4.3.12. Therefore, the only possibility is that both χ_1 and χ_2 are unramified at $\mathfrak{p}$, and hence ρ_f is unramified at $\mathfrak{p}$.

Let us suppose that the Galois representation ρ_f attached to f is absolutely irreducible, and therefore by [Car94, Théorème 2], there exists a free of rank two $\mathbf{T}_{\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)}$-module $\mathcal{M}$ with a continuous action of G_F such that the G_F action on $\mathcal{M}$ induces a G_F-equivariant isomorphism

$$\mathcal{M} \otimes_{\mathcal{O}} K \simeq \prod_{g\in\mathcal{N}} V(g),$$

where $V(g)$ denotes the $K[D_{\mathfrak{p}}]$-module corresponding to the Galois representation attached to g. The $T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)}$-eigenvalue for any $g \in \mathcal{N}$ is an element of $\mathcal{O}^\times$, reducing to $\alpha_{\mathfrak{p}}^\circ$ modulo ϖ, i.e. each $g \in \mathcal{N}$ is nearly-ordinary at $\mathfrak{p}$. Therefore, by Theorem 4.3.8, one has a short exact sequence of $K[D_{\mathfrak{p}}]$-modules

$$0 \longrightarrow V(g)^+ \longrightarrow V(g) \longrightarrow V(g)^- \longrightarrow 0,$$

where $V(g)^+$ and $V(g)^-$ have dimension 1 over K. Moreover, $D_{\mathfrak{p}}$ acts on $V(g)^-/\varpi V(g)^-$ via an unramified character mapping $\mathrm{Frob}_{\mathfrak{p}}$ to $\alpha_{\mathfrak{p}}^\circ \in \mathbb{F}$. Now, let $\mathcal{M}^+ := \mathcal{M} \cap \prod_{g\in\mathcal{N}} V(g)^+$ and let $\mathcal{M}^- := \mathrm{Im}\left(\mathcal{M} \to \prod_{g\in\mathcal{N}} V(g)^-\right)$. Then the above exact sequence induces a short exact sequence of $\mathbf{T}_{\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)}[D_{\mathfrak{p}}]$-modules

$$0 \longrightarrow \mathcal{M}^+ \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}^- \longrightarrow 0.$$

Reducing modulo the maximal ideal $\mathbf{m}$ gives a short exact sequence of $\mathbb{F}[D_{\mathfrak{p}}]$-modules

$$\mathcal{M}^+/\mathbf{m}\mathcal{M}^+ \longrightarrow \mathcal{M}/\mathbf{m}\mathcal{M} \longrightarrow \mathcal{M}^-/\mathbf{m}\mathcal{M}^- \longrightarrow 0.$$

To prove that ρ_f is unramified at $\mathfrak{p}$, it now suffices to show that the $\mathbb{F}[D_{\mathfrak{p}}]$-module $\mathcal{M}/\mathbf{m}\mathcal{M}$ is isomorphic to its unramified quotient $\mathcal{M}^-/\mathbf{m}\mathcal{M}^-$.
Let us then study the $\mathbb{F}[D_{\mathfrak{p}}]$-module $\mathcal{M}^-/\mathbf{m}\mathcal{M}^-$. Since $\mathcal{M}^-$ is not 0, by Nakayama's lemma for the local $\mathcal{O}$-algebra $\mathbf{T}_{\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)}$, the $\mathbb{F}[D_{\mathfrak{p}}]$-module $\mathcal{M}^-/\mathbf{m}\mathcal{M}^-$ is not the 0 module. In particular, as an $\mathbb{F}$-vector space it has either dimension 1 or 2.
Suppose that $\dim_{\mathbb{F}} \mathcal{M}^-/\mathbf{m}\mathcal{M}^- = 1$, then Nakayama's lemma produces a surjective homomorphism $\mathbf{T}_{\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)} \twoheadrightarrow \mathcal{M}^-$ as $\mathbf{T}_{\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)}$-modules. However, they have the same rank over $\mathcal{O}$ and therefore $\mathcal{M}^-$ is a free $\mathbf{T}_{\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)}$-module of rank 1. In particular, by Theorem 4.3.8 the uniformizer $\varpi_{\mathfrak{p}}$ acts via local class field theory on $\mathcal{M}^-$ via an element $U \in \left(\mathbf{T}_{\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)}\right)^\times$. Since for every $g \in \mathcal{N}$ the eigenvalue of U on g is the unique unit root of the Hecke polynomial $X^2 - T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)}X + \langle\mathfrak{p}\rangle_1 \prod_{\tau\in\Sigma_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{k_\tau+r\kappa_{\mathrm{ex},\tau}-1}$ on g, one has that

$$\begin{aligned} \mathbf{T}_{\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)} &\longrightarrow \mathbf{T}_{\mathbb{F},\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)} \\ U &\longmapsto T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)}, \end{aligned}$$

which implies that $T_{\mathfrak{p}}^{\circ,(k+\kappa_{\mathrm{ex}},1)} \in \mathbf{T}_{\mathbb{F},\mathbf{m}}^{(k+r\kappa_{\mathrm{ex}},1)}$, but this contradicts Lemma 4.3.13.
Therefore $\dim_{\mathbb{F}} \mathcal{M}^-/\mathbf{m}\mathcal{M}^- = 2$, which implies that $\mathcal{M}/\mathbf{m}\mathcal{M} \simeq \mathcal{M}^-/\mathbf{m}\mathcal{M}^-$. Let us now look at the action of $\mathrm{Frob}_{\mathfrak{p}}$ on $\mathcal{M}/\mathbf{m}\mathcal{M}$. We know that $\varpi_{\mathfrak{p}}$ acts on each $V(g)^-$ via the unit root $\alpha(g,\mathfrak{p})$ of the polynomial $X^2 - \widetilde{\alpha_{\mathfrak{p}}^{\circ}} X + \epsilon(\mathfrak{p}) \prod_{\tau\in\Sigma_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{r\kappa_{\mathrm{ex},\tau}}$ and therefore $\alpha(g,\mathfrak{p})$ reduces to $\alpha_{\mathfrak{p}}^{\circ}$ modulo ϖ and that $\mathrm{Frob}_{\mathfrak{p}}$ acts on each $V(g)^-/\mathbf{m}V(g)^-$ via $\alpha_{\mathfrak{p}}^{\circ}$. Moreover, since $\mathcal{M}^-/\mathbf{m}\mathcal{M}^-$ is a quotient of two lattices in $\prod_{g\in\mathcal{N}} V(g)^-$, $\mathrm{Frob}_{\mathfrak{p}}$ acts on $\mathcal{M}/\mathbf{m}\mathcal{M}$ via a matrix $\begin{pmatrix} \alpha_{\mathfrak{p}}^{\circ} & * \\ 0 & \alpha_{\mathfrak{p}}^{\circ} \end{pmatrix}$, which implies that $\mathrm{tr}\left(\rho_f(\mathrm{Frob}_{\mathfrak{p}})\right) = 2\alpha_{\mathfrak{p}}^{\circ} = \lambda^{\circ}(f,\mathfrak{p})$. This completes the proof.

4.4 Future Prospects

In this paragraph we illustrate future possible directions of the work of this thesis. Recall that K is a finite extensions of $\mathbb{Q}_p$, containing the images of all embeddings of F in $\bar{\mathbb{Q}}_p$, and that we denote with $\mathcal{O}$ its valuation ring, ϖ a uniformizer and $\mathbb{F} = \mathcal{O}/\varpi$. We denote then

$$\mathcal{M}_{k,\mathsf{w}}(\mathfrak{n}; K/\mathcal{O}) := \varinjlim_{m} \mathcal{M}_{k,\mathsf{w}}\left(\mathfrak{n}; \frac{1}{\varpi^m}\mathcal{O}/\mathcal{O}\right) \simeq \varinjlim_{m} \mathcal{M}_{k,\mathsf{w}}(\mathfrak{n}; \mathcal{O}/\varpi^m\mathcal{O})\,. \tag{4.7}$$

For any paritious weight (k,w), the R-module of Hilbert modular forms $\mathcal{M}_{k,\mathsf{w}}(\mathfrak{n};R)$ is equipped with a commuting family of Hecke operators $T_{\mathfrak{q}}$ and normalized Diamond operators $\langle\mathfrak{q}\rangle_{\mathsf{w}}$ (see Section 3.1) for any prime ideal $\mathfrak{q} \subset \mathcal{O}_F$ not dividing $p\mathfrak{n}$. We then define the *paritious weight* (k,w) *Hecke algebra* as

$$\mathbf{T}^{(k,\mathsf{w})} := \mathrm{im}\left(\mathcal{O}[T_{\mathfrak{q}}, \langle\mathfrak{q}\rangle_{\mathsf{w}}]_{\mathfrak{q}\nmid p\mathfrak{n}} \to \mathrm{End}_{\mathcal{O}}\left(\mathcal{M}_{k,\mathsf{w}}(\mathfrak{n};K/\mathcal{O})\right)\right)\,. \tag{4.8}$$

Moreover, as detailed in Chapter 3, we recall that Emerton-Reduzzi-Xiao have constructed in [ERX17a] the Hecke operator $T_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ in $\mathcal{O}_F$ above p.

Our future goal is to show the following generalization to non-parallel paritious weight 1 Hilbert modular forms of results of Dimitrov-Wiese ([DW18, Theorem 1.1]), Deo-Dimitrov-Wiese ([DDW19]) and of Emerton-Reduzzi-Xiao ([ERX17a, Theorem 1.1]).

Expected Theorem 4.4.1. *Let $\mathfrak{p}|p$ be a fixed prime in $\mathcal{O}_F$, and let $(k,1)\in\mathbb{Z}^{\Sigma}\times\mathbb{Z}$ be a paritious weight such that $k_\tau = 1$ for all $\tau\in\Sigma_{\mathfrak{p}}$. Then there exists a $\mathbf{T}^{(k,1)}$-valued pseudo-representation $P^{(k,1)}$ of G_F of degree 2 which is unramified at all primes $\mathfrak{q}$ not dividing $p\mathfrak{n}$ and also at $\mathfrak{q}=\mathfrak{p}$, and such that $P^{(k,1)}(\mathrm{Frob}_{\mathfrak{q}}) = (T_{\mathfrak{q}}, \langle\mathfrak{q}\rangle)$, for any such $\mathfrak{q}$.*

In particular, if the localisation of $P^{(k,1)}$ at a maximal ideal $\mathbf{m}$ of $\mathbf{T}^{(k,1)}$ is residually absolutely irreducible, then the corresponding representation

$$\rho_{\mathbf{m}} : G_F \longrightarrow \mathrm{GL}_2\left(\mathbf{T}_{\mathbf{m}}^{(k,1)}\right)$$

exists and is unramified at all primes $\mathfrak{q}$ not dividing $p\mathfrak{n}$ and at $\mathfrak{q}=\mathfrak{p}$ and satisfies $\mathrm{tr}(\rho_{\mathbf{m}}(\mathrm{Frob}_{\mathfrak{q}})) = T_{\mathfrak{q}}$ and $\det(\rho_{\mathbf{m}}(\mathrm{Frob}_{\mathfrak{q}})) = \langle\mathfrak{q}\rangle$, for all such primes $\mathfrak{q}$.

In Lemma 4.2.5, we already showed that there exists a product of partial Hasse invariants $h_{\mathrm{ex},m}$ that we can use to bring our forms modulo ϖ^m in liftable weight and then apply Lemma 4.1.6. One would then need to use partial Theta operators, as done by Deo, Dimitrov and Wiese in [DDW19] and study $\mathfrak{p}$-*ordinary* pseudo-representations, to apply the strategy of Calegari and Specter ([CS19]).

Appendix A

Some Algebraic Geometry

In this section we will recall and show some results of algebraic geometry used in this text.

A.1 The sheaf $\mathcal{L}_M$

Let A be a ring and $X \xrightarrow{f} \mathrm{Spec}(A)$ be a scheme over A. Let M be an A-module and $\mathcal{L}$ a coherent sheaf on X. As explained in [Har77], one can construct a sheaf $\widetilde{M}$ which is a $\mathcal{O}_{\mathrm{Spec}(A)}$-module, i.e. a sheaf of modules over $\mathcal{O}_{\mathrm{Spec}(A)}$. Moreover, $\widetilde{M}$ is quasi-coherent as a sheaf of modules on $\mathcal{O}_{\mathrm{Spec}(A)}$. Applying pullback, one gets $f^*\widetilde{M}$ which is a quasi-coherent $\mathcal{O}_X$-module (again this is shown in [Har77]). One can define

$$\mathcal{L}_M := \mathcal{L} \otimes_{\mathcal{O}_X} f^*\widetilde{M}$$

as the tensor product of two $\mathcal{O}_X$-modules which is a quasi-coherent $\mathcal{O}_X$-module. Now we want to see how this sheaf looks on open sets $U \subset X$.

Claim A.1.1. *For any open $U \subset X$, $\mathcal{L}_M(U) \simeq \mathcal{L}(U) \otimes_A M$.*

Proof. Recall that for V an open subset of $\mathrm{Spec}(A)$

$$\widetilde{M}(V) = \mathcal{O}_{\mathrm{Spec}(A)}(V) \otimes_A M$$

and for U an open subset of X

$$f^*\widetilde{M}(U) = f^{-1}\widetilde{M}(U) \otimes_{f^{-1}\mathcal{O}_{\mathrm{Spec}(A)}(U)} \mathcal{O}_X(U)\ .$$

Let's first try to calculate $f^{-1}\widetilde{M}(U)$.

$$\begin{aligned}
f^{-1}\widetilde{M}(U) &= \varinjlim_{V \supset f(U)} \widetilde{M}(V) \\
&= \varinjlim_{V \supset f(U)} \Big(\mathcal{O}_{\mathrm{Spec}(A)}(V) \otimes_A M\Big) \\
&\simeq \Big(\varinjlim_{V \supset f(U)} \mathcal{O}_{\mathrm{Spec}(A)}(V)\Big) \otimes_A M \\
&= A \otimes_A M \simeq M
\end{aligned}$$

Therefore, we have that

$$f^*\widetilde{M}(U) \simeq M \otimes_A \mathcal{O}_X(U) \; .$$

Using the definition of the tensor product of sheaves, one gets that

$$\begin{aligned} \mathcal{L}_M(U) &= \mathcal{L}(U) \otimes_{\mathcal{O}_X(U)} \Big(M \otimes_A \mathcal{O}_X(U) \Big) \\ &\simeq \mathcal{L}(U) \otimes_A M \; . \end{aligned}$$

□

Therefore, one can see the sheaf $\mathcal{L}_M$ as $\mathcal{L} \otimes_A M$, which is the definition used in the text. Moreover, one now knows that it's a quasi-coherent sheaf on X.
If A is a local ring, which is the case for the main text, for every $x \in X$ one gets that $(\mathcal{L}_M)_x = \mathcal{L}_x \otimes_A M$. In fact, by known results of algebraic geometry, one would have $(\mathcal{L}_M)_x = \mathcal{L}_x \otimes_{A_{f(x)}} M_{f(x)}$, but $A_{f(x)} = A$ since A is already a local ring, and therefore also $M_{f(x)} = M$.

A.2 Torsion in Cohomology Rings

Recall that X is a modular curve over $\mathrm{Spec}(\mathcal{O})$ and ω its sheaf of differential. By the previous section, the sheaf $\omega_{K/\mathcal{O}}$ is well defined. One can see that it can be identified with the direct limit

$$\varinjlim_m \omega_{\mathcal{O}/\varpi^m\mathcal{O}} \; .$$

Our interest is now to show that modular forms with coefficients in $\mathcal{O}/\varpi^m$ can be identified with the ϖ^m-torsion of modular forms with coefficients in $K/\mathcal{O}$.

Proposition A.2.1. *There is a natural isomorphism*

$$\mathrm{H}^0(X, \omega_{\mathcal{O}/\varpi^m\mathcal{O}}) \simeq \mathrm{H}^0(X, \omega_{K/\mathcal{O}})[\varpi^m]$$

Proof. To show this isomorphism we will proceed in different steps.

1. multiplication by ϖ^m is a morphism of sheaves.
   Let us consider the map
   $$\cdot\varpi^m : \omega_{K/\mathcal{O}} \to \omega_{K/\mathcal{O}} \; ,$$
   defined on open sets $U \subset X$ by
   $$\begin{aligned} \cdot\varpi^m(U) : \omega_{K/\mathcal{O}}(U) &\longrightarrow \omega_{K/\mathcal{O}}(U) \\ x &\longmapsto x\varpi^m \end{aligned}$$
   as a homomorphism of $\mathcal{O}$-modules. To show that it's a morphism of pre-sheaf, one has to check that this map commutes with restriction maps. Let $V \subset U$ be affine open subsets of X and $\alpha_{UV} : \omega_{K/\mathcal{O}}(U) \to \omega_{K/\mathcal{O}}(V)$ the restriction map for $\omega_{K/\mathcal{O}}$. One then considers the following diagram
   $$\begin{array}{ccc} \omega_{K/\mathcal{O}}(U) & \xrightarrow{\cdot\varpi^m} & \omega_{K/\mathcal{O}}(U) \\ \Big\downarrow{\scriptstyle \alpha_{UV}} & & \Big\downarrow{\scriptstyle \alpha_{UV}} \\ \omega_{K/\mathcal{O}}(V) & \xrightarrow{\cdot\varpi^m} & \omega_{K/\mathcal{O}}(V) \end{array}$$

By the previous section we know that $\omega_{K/\mathcal{O}}(U) \simeq \omega(U) \otimes_{\mathcal{O}} K/\mathcal{O}$ and since U is an affine open, this quantity can actually be seen as $M \otimes_{\mathcal{O}} K/\mathcal{O}$, where M is an $\mathcal{O}$-module. The same can be done for V, i.e. one gets $\omega_{K/\mathcal{O}}(V) \simeq N \otimes_{\mathcal{O}} K/\mathcal{O}$ for some $\mathcal{O}$-module N. In particular $\alpha_{UV} = \alpha_{MN} \otimes \mathrm{id}$, where $\alpha_{MN} : M \to N$ is a morphism of $\mathcal{O}$-modules. The diagram above becomes

$$\begin{tikzcd} M \otimes_{\mathcal{O}} K/\mathcal{O} \arrow[r, "\cdot\varpi^m"] \arrow[d, "\alpha_{MN}\otimes\mathrm{id}"] & M \otimes_{\mathcal{O}} K/\mathcal{O} \arrow[d, "\alpha_{MN}\otimes\mathrm{id}"] \\ N \otimes_{\mathcal{O}} K/\mathcal{O} \arrow[r, "\cdot\varpi^m"] & N \otimes_{\mathcal{O}} K/\mathcal{O} \end{tikzcd}$$

which is clearly commutative.

2. $\omega_{\mathcal{O}/\varpi^m\mathcal{O}} \simeq \omega_{K/\mathcal{O}}[\varpi^m]$.
Now we want to show that $\omega_{\mathcal{O}/\varpi^m\mathcal{O}} \simeq \omega_{K/\mathcal{O}}[\varpi^m]$ as sheaves, which we will prove on the stalks. Let us first recall that $K/\mathcal{O}[\varpi^m] = \mathcal{O}/\varpi^m\mathcal{O}$. Recall that $\omega_{K/\mathcal{O}}[\varpi^m]$ is defined as the sheaf kernel of the multiplication by ϖ^m; and that given a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves, the stalk at a point x of the kernel presheaf is the kernel of φ_x. Looking at the previous section, one can see that $\omega_{K/\mathcal{O}}[\varpi^m] = (\omega \otimes_{\mathcal{O}} K/\mathcal{O})[\varpi^m]$. Since ω is an invertible sheaf, its stalks are free modules, therefore flat. By the previous section we know that the stalk of $\omega_{K/\mathcal{O}}$ at a point $x \in X$ is isomorphic to the tensor product of the stalk of ω at x with $K/\mathcal{O}$. In other words, $(\omega_{K/\mathcal{O}})_x \simeq \omega_x \otimes K/\mathcal{O}$. So, if we take the short exact sequence

$$0 \to K/\mathcal{O}[\varpi^m] \simeq \mathcal{O}/\varpi^m\mathcal{O} \to K/\mathcal{O} \xrightarrow{\cdot\varpi^m} (\varpi^m)K/\mathcal{O} \ \to 0$$

tensoring with the flat module ω_x, we still get an exact sequence:

$$0 \to \omega_x \otimes K/\mathcal{O}[\varpi^m] \to \omega_x \otimes K/\mathcal{O} \xrightarrow{\cdot\varpi^m} \omega_x \otimes (\varpi^m)K/\mathcal{O} \ \to 0$$

Therefore

$$\begin{aligned} \big(\omega_x \otimes K/\mathcal{O}\big)[\varpi^m] &\simeq \omega_x \otimes K/\mathcal{O}[\varpi^m] \\ &\simeq \omega_x \otimes \mathcal{O}/\varpi^m\mathcal{O} \\ &= \big(\omega_{\mathcal{O}/\varpi^m\mathcal{O}}\big)_x \end{aligned}$$

which means that at every $x \in X$, $\big(\omega_{\mathcal{O}/\varpi^m\mathcal{O}}\big)_x \simeq (\omega_{K/\mathcal{O}}[\varpi^m])_x$, which gives $\omega_{\mathcal{O}/\varpi^m\mathcal{O}} \simeq \omega_{K/\mathcal{O}}[\varpi^m]$.

3. $\mathrm{H}^0(X, \omega_{K/\mathcal{O}}[\varpi^m]) \simeq \mathrm{H}^0(X, \omega_{K/\mathcal{O}})[\varpi^m]$ Recall that the cohomology of level 0 is just the global section. One has that $\mathrm{H}^0(X, \omega_{K/\mathcal{O}}[\varpi^m]) = \omega_{K/\mathcal{O}}[\varpi^m](X)$ and $\mathrm{H}^0(X, \omega_{K/\mathcal{O}})[\varpi^m] = \omega_{K/\mathcal{O}}(X)[\varpi^m]$. Looking at the exact sequence of sheaves

$$0 \to \omega_{K/\mathcal{O}}[\varpi^m] \to \omega_{K/\mathcal{O}} \xrightarrow{\cdot\varpi^m} \omega_{K/\mathcal{O}}$$

and conjugating it for X

$$0 \to \omega_{K/\mathcal{O}}[\varpi^m](X) \to \omega_{K/\mathcal{O}}(X) \xrightarrow{\cdot\varpi^m} \omega_{K/\mathcal{O}}(X)$$

one gets that $\omega_{K/\mathcal{O}}[\varpi^m](X)$ is isomorphic to the kernel of the map $\omega_{K/\mathcal{O}}(X) \xrightarrow{\cdot\varpi^m} \omega_{K/\mathcal{O}}(X)$ which is exactly $\omega_{K/\mathcal{O}}(X)[\varpi^m]$.

□

Appendix B

Descent

In this brief appendix, we will recall some results of Descent theory that we apply in the thesis.

Definition B.0.1. Let S be a scheme and let $f : X \to Y$ be a morphism of S-schemes. We say that a quasi-coherent $\mathcal{O}_X$ module $\mathcal{F}$ *descends* to Y, if there exists an $\mathcal{O}_Y$-module $\mathcal{G}$ such that

$$f^*\mathcal{G} \simeq \mathcal{F} \; .$$

In general, the theory of descent studies the equivalence of fibered categories. Here we will recall some results that can be applied in our case. We are particularly interested in two specific case:

1. When Y is the quotient variety X/G by an (abelian) group G and f is a finite étale covering;
2. When X and Y are respectively the toroidal and minimal compactification of a modular variety and f is the corresponding map between the two compactifications (see Section 2.1.4 for more details).

B.1 Finite descent

The following is [Sta18, Lemma 35.6.2]

Lemma B.1.1. *Let $\pi : X \to Y$ be a surjective finite étale morphism of S-schemes. Let G be a finite group together with a group homomorphism $G^{opp} \simeq \mathrm{Aut}_Y(X)$, mapping $\sigma \mapsto f_\sigma$, such that the map*

$$\begin{aligned} G \times X &\to X \times_Y X \\ (\sigma, x) &\mapsto (x, f_\sigma(x)) \end{aligned}$$

is an isomorphism. Then The category of quasi-coherent $\mathcal{O}_Y$-modules is equivalent to the category of systems $(\mathcal{F}, (\varphi_\sigma)_{\sigma\in G})$ where

(i) $\mathcal{F}$ *is a quasi-coherent $\mathcal{O}_X$-module;*

(ii) $\varphi_\sigma : \mathcal{F} \to f_\sigma^*\mathcal{F}$ *is an isomorphism of $\mathcal{O}_X$-modules;*

(iii) $\varphi_{\sigma\tau} = f_\sigma^*\varphi_\tau \circ \varphi_\sigma$ *for all $\sigma, \tau \in G$.*

Let us recall that in our situation, we have provided the line bundles $\dot\omega_\tau$ and $\dot\delta_\tau$ defined over $\mathrm{Y}_{\mathfrak{c}}$ with an E-action, mapping respectively a local section s to $\tau(\varepsilon)^{-1/2}[\varepsilon]^*s$ and a local section s to $\tau(\varepsilon)^{-1}[\varepsilon]^*s$. This correspond to the isomoprhism φ_σ of the above lemma. It is easy to check that these maps satisfy the conditions of Lemma B.1.1, since the group E is abelian, and therefore they descend to line bundles ω_τ and δ_τ over the quotient variety Sh.

In particular, for a finite abelian group G, one can see that if $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_Y$-module corresponding to a system $(\mathcal{F}, (\varphi_\sigma)_{\sigma\in G})$, then

$$\mathrm{H}^0(Y, \mathcal{G}) = \mathrm{H}^0(X, \mathcal{F})^G, \tag{B.1}$$

which is the set of global sections which are invariant under the action of G. Therefore, in our case, the set of *Hilbert modular forms* $\mathrm{H}^0(\mathrm{Sh}, \omega^{k,\ell})$ corresponds to the subset of elements in $\mathrm{H}^0(\mathrm{Y}, \dot\omega^{k,\ell})$ which are invariant under the action of the group E.

B.2 Descent for the compactifications

As seen in Section 2.1.4, one can construct the toroidal and the minimal compactification of the moduli space $\mathrm{Y}_{\mathfrak{c}}$, and moreover one has a surjectif morphism

$$\pi : \mathrm{Y}_{\mathfrak{c}}^{\mathrm{tor}} \twoheadrightarrow \mathrm{Y}_{\mathfrak{c}}^{\mathrm{min}} ,$$

which is an isomorphism on $\mathrm{Y}_{\mathfrak{c}}$.

Lemma B.2.1. *The line bundle $\dot\omega^k_{\mathfrak{c},R} = \bigotimes_{\tau\in\Sigma}(\omega^{\mathrm{tor}}_{\mathfrak{c},R,\tau})^{\otimes k_\tau}$ on $\mathrm{Y}^{\mathrm{tor}}_{\mathfrak{c},R}$ descends to a line bundle on $\mathrm{Y}^{\mathrm{min}}_{\mathfrak{c},R}$ if and only if $u^{k/2} := \prod_{\tau\in\Sigma}\tau(u)^{k_\tau/2}$ acts trivially in R, for all $u \in \mathcal{O}^\times_{F,\mathfrak{n}}$.*

In characteristic 0, Lemma B.2.1 implies that the sheaf $\dot\omega^k$ descends to the minimal compactification if and only if k is parallel (as stated in [Dim04, Théorème 8.6(vi)]).

Proof. We will prove this on formal schemes, and in particular on the formal completion at the cusps. By [Dim04, Théorème 8.6(v)], we know that the formal completion of $\mathrm{Y}^{\mathrm{tor}}_{\mathfrak{c},R}$ at the inverse image $\pi^{-1}(\mathcal{C})$ of a cusp $\mathcal{C}$ of $\mathrm{Y}_{\mathfrak{c},R}$, seen inside $\mathrm{Y}^{\mathrm{min}}_{\mathfrak{c},R}$, is $S^\wedge_{\Sigma\mathcal{C}}/\mathcal{O}^\times_{F,\mathfrak{n}} \times \mathrm{Spec}(R)$. This tells us that the sheaf $\dot\omega^k_{\mathfrak{c},R}$ will descend to an invertible sheaf on the minimal compactification $\mathrm{Y}^{\mathrm{min}}_{\mathfrak{c},R}$ if and only if $\dot\omega^k_{\mathfrak{c},R}$ can be trivialized over $S^\wedge_{\Sigma\mathcal{C}}/\mathcal{O}^\times_{F,\mathfrak{n}} \times \mathrm{Spec}(R)$. Moreover, the formal completion of $\mathrm{Y}^{\mathrm{tor}}_{\mathfrak{c},R}$ at the cusp $\mathcal{C}$ of $\mathrm{Y}_{\mathfrak{c},R}$, seen inside $\mathrm{Y}^{\mathrm{tor}}_{\mathfrak{c},R}$, is $S^\wedge_{\Sigma\mathcal{C}} \times \mathrm{Spec}(R)$. In particular, we have the following surjective morphism of formal schemes:

$$S^\wedge_{\Sigma\mathcal{C}} \times \mathrm{Spec}(R) \twoheadrightarrow S^\wedge_{\Sigma\mathcal{C}}/\mathcal{O}^\times_{F,\mathfrak{n}} \times \mathrm{Spec}(R) \tag{B.2}$$

As seen throughout the thesis (see Remark 2.3.4 and proof of Proposition 2.4.1), the sheaf $\dot\omega^k_{\mathfrak{c},R}$ can be canonically trivialized over $S^\wedge_{\Sigma\mathcal{C}} \times \mathrm{Spec}(R)$ as follows:

$$\dot\omega^k_{\mathfrak{c},R}|_{S^\wedge_{\Sigma\mathcal{C}}\times\mathrm{Spec}(R)} \simeq (\mathfrak{a}\otimes\mathcal{O})^k \otimes_{\mathcal{O}} R \otimes_{\mathcal{O}} \mathcal{O}_{S^\wedge_{\Sigma\mathcal{C}}} ,$$

where $\mathcal{C} = (\mathfrak{a}, \mathfrak{b}, H, i, j, \lambda, \alpha)$ (see Definition 2.3.1). Moreover, an element $u \in \mathcal{O}^\times_{F,\mathfrak{n}}$ acts on this sheaf via multiplication by $u^{k/2} = \prod_{\tau\in\Sigma}\tau(u)^{k_\tau/2}$. In particular,

$$\mathrm{H}^0(S^\wedge_{\Sigma\mathcal{C}}/\mathcal{O}^\times_{F,\mathfrak{n}}\times\mathrm{Spec}(R), \dot\omega^k_{\mathfrak{c},R}) = \left\{ \sum_{\xi\in X_+\cup\{0\}} a_\xi q^\xi : a_\xi \in (\mathfrak{a}\otimes\mathcal{O})^k\otimes_{\mathcal{O}}R, a_{u\xi} = u^2 a_\xi \text{ for all } u \in \mathcal{O}^\times_{F,\mathfrak{n}} \right\},$$

which is not a priori of rank 1 as a projective module. Therefore the pullback of $\dot{\omega}^k_{\mathfrak{c},R}$ to $S^{\wedge}_{\Sigma^{\mathcal{C}}} \times \operatorname{Spec}(R)$ via the map in Equation B.2 will be canonically trivial if and only if $u^{k/2}$ acts trivially in R. $\square$

We have applied this Lemma in the proof of Lemma 4.1.2 to descend the sheaf $\dot{\omega}_{\mathbb{F}}^{(\mathrm{ex},\,0)}$ to an ample invertible sheaf over the minimal compactification.

Appendix C

Extra Calculations with q-expansions

We collect in this appendix proofs by computations on q-expansions of well known relations between the various operators defined throughout the thesis.

C.1 Proofs by calculations on q-expansions

We will use here the Notation of Chapter 4.

Proposition C.1.1. *Let $f \in \mathcal{M}_{k,\mathsf{w}}(\mathfrak{n}; R)$. Then $T_{\mathfrak{r}}(T_{\mathfrak{q}} f) = T_{\mathfrak{q}}(T_{\mathfrak{r}} f)$ for all $\mathfrak{r}, \mathfrak{q}$ prime ideals of $\mathcal{O}_F$, coprime with $\mathfrak{n}p$.*

Proof. We prove it on q-expansions for non-normalized Hecke operators, using Equation (3.19). We want to show that for any $\mathfrak{c} \in \mathfrak{C}$ and $\xi \in \mathfrak{c}_+$

$$a^\circ_\xi \left(\left(T_{\mathfrak{r}}(T_{\mathfrak{q}} f)\right)_{\mathfrak{c}} \right) = a^\circ_\xi \left(\left(T_{\mathfrak{q}}(T_{\mathfrak{r}} f)\right)_{\mathfrak{c}} \right) \ . \tag{C.1}$$

Let us start by applying Equation (3.18) to the left hand side of the above equation. Recall that we take $\mathfrak{c}', \mathfrak{c}'' \in \mathfrak{C}$ and $\alpha, \beta \in F_+$ such that $\alpha\mathfrak{c}' = \mathfrak{c}\mathfrak{r}$ and $\beta\mathfrak{c}'' = \mathfrak{c}\mathfrak{r}^{-1}$, then we have

$$\begin{aligned} a^\circ_\xi \left(\left(T_{\mathfrak{r}}(T_{\mathfrak{q}} f)\right)_{\mathfrak{c}} \right) &= \mathrm{Nm}(\mathfrak{r})^{\mathsf{w}-1} \alpha^{-\ell} a^\circ_{\alpha^{-1}\xi}\big((\langle \mathfrak{r} \rangle_{\mathsf{w}} T_{\mathfrak{q}} f)_{\mathfrak{c}'} \big) + \beta^{-\ell} a^\circ_{\beta^{-1}\xi}\big((T_{\mathfrak{q}} f)_{\mathfrak{c}''} \big) \\ &= \mathrm{Nm}(\mathfrak{r})^{\mathsf{w}-1} \alpha^{-\ell} \Big(\mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1} \tilde{\alpha}^{-\ell} a^\circ_{(\tilde{\alpha}\alpha)^{-1}\xi}\big((\langle \mathfrak{q} \rangle_{\mathsf{w}} \langle \mathfrak{r} \rangle_{\mathsf{w}} f)_{\tilde{\mathfrak{c}}'} \big) + \tilde{\beta}^{-\ell} a^\circ_{(\tilde{\beta}\alpha)^{-1}\xi}\big(\langle \mathfrak{r} \rangle_{\mathsf{w}} f_{\tilde{\mathfrak{c}}''} \big) \Big) \\ &+ \beta^{-\ell} \Big(\mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1} \tilde{\tilde{\alpha}}^{-\ell} a^\circ_{(\tilde{\tilde{\alpha}}\beta)^{-1}\xi}\big((\langle \mathfrak{q} \rangle_{\mathsf{w}} f)_{\tilde{\tilde{\mathfrak{c}}}'} \big) + \tilde{\tilde{\beta}}^{-\ell} a^\circ_{(\tilde{\tilde{\beta}}\beta)^{-1}\xi}\big(f_{\tilde{\tilde{\mathfrak{c}}}''} \big) \Big) , \end{aligned}$$

where $\mathfrak{c}'\mathfrak{q} = \tilde{\alpha}\tilde{\mathfrak{c}}'$, $\mathfrak{c}'\mathfrak{q}^{-1} = \tilde{\beta}\tilde{\mathfrak{c}}''$ $\mathfrak{c}''\mathfrak{q} = \tilde{\tilde{\alpha}}\tilde{\tilde{\mathfrak{c}}}'$ and $\mathfrak{c}''\mathfrak{q}^{-1} = \tilde{\tilde{\beta}}\tilde{\tilde{\mathfrak{c}}}''$, for $\tilde{\mathfrak{c}}', \tilde{\mathfrak{c}}'', \tilde{\tilde{\mathfrak{c}}}', \tilde{\tilde{\mathfrak{c}}}'' \in \mathfrak{C}$ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}} \in F_+$. Rearranging the terms, one has that

$$\begin{aligned} a^\circ_\xi \left(\left(T_{\mathfrak{r}}(T_{\mathfrak{q}} f)\right)_{\mathfrak{c}} \right) &= \big(\mathrm{Nm}(\mathfrak{r}) \, \mathrm{Nm}(\mathfrak{q}) \big)^{\mathsf{w}-1} (\tilde{\alpha}\alpha)^{-\ell} a^\circ_{(\tilde{\alpha}\alpha)^{-1}\xi}\big((\langle \mathfrak{r} \rangle_{\mathsf{w}} \langle \mathfrak{q} \rangle_{\mathsf{w}} f)_{\tilde{\mathfrak{c}}'} \big) \\ &+ \mathrm{Nm}(\mathfrak{r})^{\mathsf{w}-1} (\tilde{\beta}\alpha)^{-\ell} \, a^\circ_{(\tilde{\beta}\alpha)^{-1}\xi}\big((\langle \mathfrak{r} \rangle_{\mathsf{w}} f)_{\tilde{\mathfrak{c}}''} \big) \\ &+ \mathrm{Nm}(\mathfrak{q})^{\mathsf{w}-1} (\tilde{\tilde{\alpha}}\beta)^{-\ell} a^\circ_{(\tilde{\tilde{\alpha}}\beta)^{-1}\xi}\big((\langle \mathfrak{q} \rangle_{\mathsf{w}} f)_{\tilde{\tilde{\mathfrak{c}}}'} \big) \\ &+ (\tilde{\tilde{\beta}}\beta)^{-\ell} a^\circ_{(\tilde{\tilde{\beta}}\beta)^{-1}\xi}\big(f_{\tilde{\tilde{\mathfrak{c}}}''} \big) \ . \end{aligned}$$

Let us now look at the right hand side of Equation (C.1). Let $\bar{\mathfrak{c}}', \bar{\mathfrak{c}}'' \in \mathfrak{C}$ and $\bar{\alpha}, \bar{\beta} \in F_+$ such that $\bar{\alpha}\bar{\mathfrak{c}}' = \mathfrak{c}\mathfrak{q}$ and $\bar{\beta}\bar{\mathfrak{c}}'' = \mathfrak{c}\mathfrak{q}^{-1}$. Then one has that $\bar{\mathfrak{c}}'\mathfrak{r} = \frac{\tilde{\alpha}\alpha}{\bar{\alpha}}\tilde{\mathfrak{c}}'$, $\bar{\mathfrak{c}}'\mathfrak{r}^{-1} = \frac{\tilde{\tilde{\alpha}}\beta}{\bar{\alpha}}\tilde{\tilde{\mathfrak{c}}}'$, $\bar{\mathfrak{c}}''\mathfrak{r} = \frac{\tilde{\beta}\alpha}{\bar{\beta}}\tilde{\mathfrak{c}}''$ and $\bar{\mathfrak{c}}''\mathfrak{r}^{-1} = \frac{\tilde{\tilde{\beta}}\beta}{\bar{\beta}}\tilde{\tilde{\mathfrak{c}}}''$. Applying Equation (3.19) twice with these representatives, one gets that

$$\begin{aligned} a^\circ_\xi\left(\left(T_{\mathfrak{q}}(T_{\mathfrak{r}}f)\right)_{\mathfrak{c}}\right) = {} & \left(\mathrm{Nm}(\mathfrak{r})\,\mathrm{Nm}(\mathfrak{q})\right)^{\mathtt{w}-1}(\tilde{\alpha}\alpha)^{-\ell}a^\circ_{(\tilde{\alpha}\alpha)^{-1}\xi}\big((\langle\mathfrak{r}\rangle_{\mathtt{w}}\langle\mathfrak{q}\rangle_{\mathtt{w}}f)_{\tilde{\mathfrak{c}}'}\big) \\ & + \mathrm{Nm}(\mathfrak{q})^{\mathtt{w}-1}(\tilde{\tilde{\alpha}}\beta)^{-\ell}a^\circ_{(\tilde{\tilde{\alpha}}\beta)^{-1}\xi}\big((\langle\mathfrak{q}\rangle_{\mathtt{w}}f)_{\tilde{\tilde{\mathfrak{c}}}'}\big) \\ & + \mathrm{Nm}(\mathfrak{r})^{\mathtt{w}-1}(\tilde{\beta}\alpha)^{-\ell}a^\circ_{(\tilde{\beta}\alpha)^{-1}\xi}\big((\langle\mathfrak{r}\rangle_{\mathtt{w}}f)_{\tilde{\mathfrak{c}}''}\big) \\ & + (\tilde{\tilde{\beta}}\beta)^{-\ell}a^\circ_{(\tilde{\tilde{\beta}}\beta)^{-1}\xi}(f_{\tilde{\tilde{\mathfrak{c}}}''})\ , \end{aligned}$$

and therefore the commutativity of the operators. □

The following Proposition is a direct consequence of Proposition 4.3.6(2). However, we show it here via computations on q-expansions.

Proposition C.1.2. *Let $f \in \mathcal{S}_{k,1}(\mathfrak{n}, \epsilon; \mathbb{F})$ be an eigenform for all $T_{\mathfrak{q}}$, for $\mathfrak{q}$ a prime of $\mathcal{O}_F$, $q \nmid \mathfrak{n}p$, and suppose that it is also an eigenform for $T^{\circ,(k,1)}_{\mathfrak{p}}$ with eigenvalue $\lambda^\circ(f,\mathfrak{p})$. Then the $T^{\circ,(k+\kappa_{\mathrm{ex}},1)}_{\mathfrak{p}}$ operator is such that $\big(T^{\circ,(k+\kappa_{\mathrm{ex}},1)}_{\mathfrak{p}}\big)^2(h_{\mathrm{ex}}f) - \lambda^\circ(f,\mathfrak{p})T^{\circ,(k+\kappa_{\mathrm{ex}},1)}_{\mathfrak{p}}(h_{\mathrm{ex}}f) + \epsilon(\mathfrak{p})(h_{\mathrm{ex}}f) = 0$.*

Proof. Let us start by looking at the q-expansion of $h_{\mathrm{ex}}f$ for any $\mathfrak{c} \in \mathfrak{C}$.

$$\big(h_{\mathrm{ex}}f\big)_{\mathfrak{c}} = \sum_{\xi\in\mathfrak{c}_+} a^\circ_\xi\big((h_{\mathrm{ex}}f)_{\mathfrak{c}}\big)q^\xi = \sum_{\xi\in\mathfrak{c}_+} a^\circ_\xi(f_{\mathfrak{c}})q^\xi\ .$$

Now by Equation 4.3, one has that for $\xi \in \mathfrak{c}_+$

$$\begin{aligned} a^\circ_\xi\big((T^{\circ,(k+\kappa_{\mathrm{ex}},1)}_{\mathfrak{p}}h_{\mathrm{ex}}f)_{\mathfrak{c}}\big) &= \varpi_{\mathfrak{p}}^{(p-1)(\mathtt{Nt}_{\mathfrak{p}}+\mathtt{ex}_{\mathfrak{p}})}\beta^{(k-\mathtt{t})/2+\kappa_{\mathrm{ex}}/2}a^\circ_{\beta^{-1}\xi}\big((h_{\mathrm{ex}}f)_{\mathfrak{c}''}\big) \\ &= \varpi_{\mathfrak{p}}^{(p-1)(\mathtt{Nt}_{\mathfrak{p}}+\mathtt{ex}_{\mathfrak{p}})}\beta^{\kappa_{\mathrm{ex}}/2}\beta^{(k-\mathtt{t})/2}a^\circ_{\beta^{-1}\xi}(f_{\mathfrak{c}''})\ , \end{aligned}$$

where $\mathfrak{c}'' \in \mathfrak{C}'$ and $\beta \in F_+$ are such that $\beta\mathfrak{c}'' = \mathfrak{c}\mathfrak{p}^{-1}$. Recall that $\varpi_{\mathfrak{p}}^{(p-1)(\mathtt{Nt}_{\mathfrak{p}}+\mathtt{ex}_{\mathfrak{p}})}\beta^{\kappa_{\mathrm{ex}}/2} \equiv 1 \mod \varpi$ (see proof of Proposition 4.3.3), and therefore $a^\circ_\xi\big((T^{\circ,(k+\kappa_{\mathrm{ex}},1)}_{\mathfrak{p}}h_{\mathrm{ex}}f)_{\mathfrak{c}}\big) = \beta^{(k-\mathtt{t})/2}a^\circ_{\beta^{-1}\xi}(f_{\mathfrak{c}''})$. Now let us look at the action of $T^{\circ,(k,1)}_{\mathfrak{p}}$ on f: for any $\xi \in \mathfrak{c}_+$, one has that

$$\begin{aligned} a^\circ_\xi\big((T^{\circ,(k,1)}_{\mathfrak{p}}f)_{\mathfrak{c}}\big) &= \alpha^{(k-\mathtt{t})/2}a^\circ_{\alpha^{-1}\xi}\big((\langle\mathfrak{p}\rangle_{\mathtt{w}}f)_{\mathfrak{c}'}\big) + \beta^{(k-\mathtt{t})/2}a^\circ_{\beta^{-1}\xi}(f_{\mathfrak{c}''}) \\ &= \epsilon(\mathfrak{p})\alpha^{(k-\mathtt{t})/2}a^\circ_{\alpha^{-1}\xi}(f_{\mathfrak{c}'}) + \beta^{(k-\mathtt{t})/2}a^\circ_{\beta^{-1}\xi}(f_{\mathfrak{c}''})\ , \end{aligned}$$

where $\mathfrak{c}' \in \mathfrak{C}'$ and $\alpha \in F_+$ are such that $\alpha\mathfrak{c}' = \mathfrak{c}\mathfrak{p}$. Then we can re-write the action of $T^{\circ,(k+\kappa_{\mathrm{ex}},1)}_{\mathfrak{p}}$ on $h_{\mathrm{ex}}f$ via the action of $T^{\circ,(k,1)}_{\mathfrak{p}}$ on f as follows: let $\xi \in \mathfrak{c}_+$, then

$$a^\circ_\xi\big((T^{\circ,(k+\kappa_{\mathrm{ex}},1)}_{\mathfrak{p}}h_{\mathrm{ex}}f)_{\mathfrak{c}}\big) = a^\circ_\xi\big((T^{\circ,(k,1)}_{\mathfrak{p}}f)_{\mathfrak{c}}\big) - \epsilon(\mathfrak{p})\alpha^{(k-\mathtt{t})/2}a^\circ_{\alpha^{-1}\xi}(f_{\mathfrak{c}'})$$

If we transpose this to $\xi \in \mathfrak{c}''_+$, one gets that

$$a^\circ_\xi\big((T^{\circ,(k+\kappa_{\mathrm{ex}},1)}_{\mathfrak{p}}h_{\mathrm{ex}}f)_{\mathfrak{c}''}\big) = a^\circ_\xi\big((T^{\circ,(k,1)}_{\mathfrak{p}}f)_{\mathfrak{c}''}\big) - \epsilon(\mathfrak{p})\beta^\ell a_{\beta\xi}(f_{\mathfrak{c}})\ .$$

Let us now apply a second time the $T_{\mathfrak{p}}^{\circ,(k+\kappa_{\rm ex},1)}$ operator to $h_{\rm ex}f$: let $\xi \in \mathfrak{c}_+$, then

$$a_\xi^\circ\big((T_{\mathfrak{p}}^{\circ,(k+\kappa_{\rm ex},1)}(T_{\mathfrak{p}}^{\circ,(k+\kappa_{\rm ex},1)}h_{\rm ex}f))_{\mathfrak{c}}\big) = \beta^{(k-\mathsf{t})/2}a^\circ_{\beta^{-1}\xi}\big((T_{\mathfrak{p}}^{\circ,(k+\kappa_{\rm ex},1)}h_{\rm ex}f)_{\mathfrak{c}''}\big)$$
$$= \beta^{(k-\mathsf{t})/2}a^\circ_{\beta^{-1}\xi}\big((T_{\mathfrak{p}}^{\circ,(k,1)}f)_{\mathfrak{c}''}\big) - \epsilon(\mathfrak{p})a_\xi(f_{\mathfrak{c}})\ .$$

By hypothesis f is an eigenform for $T_{\mathfrak{p}}^{\circ,(k,1)}$ of eigenvalue $\lambda^\circ(f,\mathfrak{p})$. In particular, for any $\mathfrak{c} \in \mathfrak{C}$ and for any $\xi \in \mathfrak{c}_+$, $a_\xi^\circ\big((T_{\mathfrak{p}}^{\circ,(k,1)}f)_{\mathfrak{c}}\big) = \lambda^\circ(f,\mathfrak{p})a_\xi^\circ(f_{\mathfrak{c}})$. Therefore the above equation becomes:

$$a_\xi^\circ\big((T_{\mathfrak{p}}^{\circ,(k+\kappa_{\rm ex},1)}(T_{\mathfrak{p}}^{\circ,(k+\kappa_{\rm ex},1)}h_{\rm ex}f))_{\mathfrak{c}}\big) = \lambda^\circ(f,\mathfrak{p})\beta^{(k-\mathsf{t})/2}a^\circ_{\beta^{-1}\xi}(f_{\mathfrak{c}''}) - \epsilon(\mathfrak{p})a_\xi(f_{\mathfrak{c}})$$
$$= \lambda(f,\mathfrak{p})a_\xi^\circ\big((T_{\mathfrak{p}}^{\circ,(k+\kappa_{\rm ex},1)}h_{\rm ex}f)_{\mathfrak{c}}\big) - \epsilon(\mathfrak{p})a_\xi^\circ\big((h_{\rm ex}f)_{\mathfrak{c}}\big)\ .$$

Therefore, one gets that $T_{\mathfrak{p}}^{\circ,(k+\kappa_{\rm ex},1)}$ is annihilated by $X^2 - \lambda(f,\mathfrak{p})X + \epsilon(\mathfrak{p})\,\mathrm{id}$. □

The following Proposition is used in the proof of part (3) of 4.3.6.

Proposition C.1.3. *Let $f \in \mathcal{S}_{(k,1)}(\mathfrak{n},\epsilon,\mathbb{F})$ a Hilbert modular form of partial weight 1, with parallel weight 1 over the fixed place $\mathfrak{p}$ above p. Let $\mathfrak{q}$ be an integral ideal of $\mathcal{O}_F$, coprime with $\mathfrak{n}p$. Then the operators $T_{\mathfrak{q}}$ and $V_{\mathfrak{p}}$ commute, i.e.*

$$T_{\mathfrak{q}}^{(k+\kappa_{\rm ex},1)}\big(V_{\mathfrak{p}}f\big) = V_{\mathfrak{p}}\big(T_{\mathfrak{q}}^{(k,1)}f\big)\ . \tag{C.2}$$

Proof. We will prove it on geometric q-expansion, i.e. we want to show that for any $\xi \in \mathfrak{c}_+$, where $\mathfrak{c} \in \mathfrak{C}$, one has that

$$a_\xi^\circ\Big(\big(T_{\mathfrak{q}}^{(k+\kappa_{\rm ex},1)}(V_{\mathfrak{p}}f)\big)_{\mathfrak{c}}\Big) = a_\xi^\circ\Big(\big(V_{\mathfrak{p}}(T_{\mathfrak{q}}^{(k,1)}f)\big)_{\mathfrak{c}}\Big)\ . \tag{C.3}$$

Let us start on the right hand side. Let $\mathfrak{c}' \in \mathfrak{C}$ and $\alpha \in F_+$ such that $\alpha\mathfrak{c}' = \mathfrak{c}\mathfrak{p}$. By Proposition 4.3.3,

$$a_\xi^\circ\Big(\big(V_{\mathfrak{p}}(T_{\mathfrak{q}}^{(k,1)}f)\big)_{\mathfrak{c}}\Big) = \alpha^{(k-\mathsf{t})/2}a^\circ_{\alpha^{-1}\xi}\big((T_{\mathfrak{q}}^{(k,1)}f)_{\mathfrak{c}'}\big)$$

Let $\tilde{\mathfrak{c}}', \tilde{\mathfrak{c}}'' \in \mathfrak{C}$ and $\tilde\alpha, \tilde\beta \in F_+$ such that $\mathfrak{c}'\mathfrak{q} = \tilde\alpha\tilde{\mathfrak{c}}'$ and $\mathfrak{c}'\mathfrak{q}^{-1} = \tilde\beta\tilde{\mathfrak{c}}''$. Then by Equation (3.19) and since $\mathsf{w} = 1$, one has that

$$a_\xi^\circ\Big(\big(V_{\mathfrak{p}}(T_{\mathfrak{q}}^{(k,1)}f)\big)_{\mathfrak{c}}\Big) = \alpha^{(k-\mathsf{t})/2}\Big(\epsilon(\mathfrak{q})\tilde\alpha^{(k-\mathsf{t})/2}a^\circ_{(\tilde\alpha\alpha)^{-1}\xi}(f_{\tilde{\mathfrak{c}}'}) + \tilde\beta^{(k-\mathsf{t})/2}a^\circ_{(\tilde\beta\alpha)^{-1}\xi}(f_{\tilde{\mathfrak{c}}''})\Big)$$
$$= \epsilon(\mathfrak{q})(\tilde\alpha\alpha)^{(k-\mathsf{t})/2}a^\circ_{(\tilde\alpha\alpha)^{-1}\xi}(f_{\tilde{\mathfrak{c}}'}) + (\tilde\beta\alpha)^{(k-\mathsf{t})/2}a^\circ_{(\tilde\beta\alpha)^{-1}\xi}(f_{\tilde{\mathfrak{c}}''})\ .$$

Let us now look at the left hand side of (C.3). Let $\bar{\mathfrak{c}}', \bar{\mathfrak{c}}'' \in \mathfrak{C}$ and $\bar\alpha, \bar\beta \in F_+$ be such that $\mathfrak{c}\mathfrak{q} = \bar\alpha\bar{\mathfrak{c}}'$ and $\mathfrak{c}\mathfrak{q}^{-1} = \bar\beta\bar{\mathfrak{c}}''$. Then by Equation (3.19)

$$a_\xi^\circ\Big(\big(T_{\mathfrak{q}}^{(k+\kappa_{\rm ex},1)}(V_{\mathfrak{p}}f)\big)_{\mathfrak{c}}\Big) = \epsilon(\mathfrak{q})\bar\alpha^{(k-\mathsf{t})/2}a_{\bar\alpha^{-1}\xi}\big((V_{\mathfrak{p}}f)_{\bar{\mathfrak{c}}'}\big) + \bar\beta^{(k-\mathsf{t})/2}a_{\bar\beta^{-1}\xi}\big((V_{\mathfrak{p}}f)_{\bar{\mathfrak{c}}''}\big)$$

Remark that $\bar{\mathfrak{c}}'\mathfrak{p} = \frac{\alpha\tilde\alpha}{\bar\alpha}\tilde{\mathfrak{c}}'$ and that $\bar{\mathfrak{c}}''\mathfrak{p} = \frac{\alpha\tilde\beta}{\bar\beta}\tilde{\mathfrak{c}}''$, and by Proposition 4.3.3, the above expression becomes

$$a_\xi^\circ\Big(\big(T_{\mathfrak{q}}^{(k+\kappa_{\rm ex},1)}(V_{\mathfrak{p}}f)\big)_{\mathfrak{c}}\Big) = \epsilon(\mathfrak{q})\bar\alpha^{(k-\mathsf{t})/2}\left(\left(\frac{\alpha\tilde\alpha}{\bar\alpha}\right)^{(k-\mathsf{t})/2}a_{(\alpha\tilde\alpha)^{-1}\xi}(f_{\tilde{\mathfrak{c}}'})\right)$$
$$+\ \bar\beta^{(k-\mathsf{t})/2}\left(\left(\frac{\alpha\tilde\beta}{\bar\beta}\right)^{(k-\mathsf{t})/2}a_{(\alpha\tilde\beta)^{-1}\xi}(f_{\tilde{\mathfrak{c}}'})\right)\ .$$

Therefore the left hand side

$$a_{\xi}^{\circ}\Big(\big(T_{\mathfrak{q}}^{(k+\kappa_{\mathrm{ex}},1)}(V_{\mathfrak{p}}f)\big)_{\mathfrak{c}}\Big) = \epsilon(\mathfrak{q})(\tilde{\alpha}\alpha)^{(k-\mathsf{t})/2}a_{(\tilde{\alpha}\alpha)^{-1}\xi}(f_{\tilde{\mathfrak{c}}'}) + (\tilde{\beta}\alpha)^{(k-\mathsf{t})/2}a_{(\tilde{\beta}\alpha)^{-1}\xi}(f_{\tilde{\mathfrak{c}}''}) \,,$$

coincides with the right hand side. □

This in particular means that if f is an eigenform for $T_{\mathfrak{q}}^{(k,1)}$, with eigenvalue $\lambda(f,q)$, then $V_{\mathfrak{p}}$ is also an eigenform for $T_{\mathfrak{q}}^{(k+\kappa_{\mathrm{ex}},1)}$ with same eigenvalue. In fact, $T_{\mathfrak{q}}^{(k+\kappa_{\mathrm{ex}},1)}(V_{\mathfrak{p}}f) = V_{\mathfrak{p}}(T_{\mathfrak{q}}^{(k,1)}f) = V_{\mathfrak{p}}(a(f,\mathfrak{q})f) = \lambda(f,\mathfrak{q})V_{\mathfrak{p}}f$.

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